Discrete Structures for Computer Science

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Lecture #7: Proof Techniques

Based on materials developed by Dr. Adam Lee
Today's topics

- Proof techniques
  - How can I prove an implication is true?
  - What forms can an informal proof take?

- Proof strategies
  - Which proof techniques should I try?
  - How do I find a proof without trying every proof technique?
Mathematical theorems are often stated in the form of an implication

**Example:** If $x > y$, where $x$ and $y$ are positive real numbers, then $x^2 > y^2$.

- $\forall x, y \ [(x > 0) \land (y > 0) \land (x > y) \rightarrow (x^2 > y^2)]$
- $\forall x, y \ (P(x, y) \rightarrow Q(x, y))$

We will discuss three applicable proof methods:

- Direct proof
- Proof by contraposition
- Proof by contradiction
In a **direct proof**, we prove $p \rightarrow q$ by showing that if $p$ is true, then $q$ must necessarily be true.

**Example:** Prove that if $n$ is an odd integer, then $n^3$ is an odd integer.

**Proof:**

1. Assume that $n$ is odd. That is $n = (2k + 1)$ for some integer $k$.
2. So, $n^3 = (2k + 1)^3 = (4k^2 + 4k + 1)(2k+1) = 8k^3 + 12k^2 + 6k + 1$.
3. We can factor the above to get $2(4k^3 + 6k^2 + 3k) + 1$.
4. Since the above quantity is one more than an even number, we know that $n^3$ is odd.

☐
Direct proofs are not always the easiest way to prove a given conjecture.

In this case, we can try **proof by contraposition**

How does this work?

- Recall that $p \rightarrow q \equiv \neg q \rightarrow \neg p$
- Therefore, a proof of $\neg q \rightarrow \neg p$ is also a proof of $p \rightarrow q$

Proof by contraposition is an **indirect** proof technique since we don’t prove $p \rightarrow q$ directly.

Let’s take a look at an example...
Prove: If $n$ is an integer and $n^2$ is even, then $n$ is even.

First, attempt a direct proof:

- Assume that $n^2$ is even, thus $n^2 = 2k$ for some integer $k$
- Can solve to find that $n = \sqrt{2k}$
- ... ?

Now, try proof by contraposition:

- Assume $n$ is odd, thus $n = 2k + 1$ for some integer $k$
- $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1$
- $= 2(2k^2 + 2k) + 1$
- Thus, $n^2$ is odd, and we proved $\neg \text{“n is even”} \rightarrow \neg \text{“n}^2 \text{ is even”}$, we can conclude that $\text{“n}^2 \text{ is even”} \rightarrow \text{“n is even”}$ ☐
Proof by contradiction

Given a conditional $p \rightarrow q$, the only way to reject this claim is to prove that $p \land \neg q$ is true.

In a proof by contradiction we:

1. Assume that $p \land \neg q$ is true
2. Proceed with the proof
3. If this assumption leads us to a contradiction, we can conclude that $p \rightarrow q$ is true
**Prove:** If $n$ is an integer and $3n + 2$ is odd, then $n$ is odd.

**Proof:**

- Assume the negation, that $3n + 2$ is odd, $n$ is even (i.e., $n = 2k$).
- $3n + 2 = 3(2k) + 2 = 6k + 2 = 2(3k + 1)$.
- The above statement tells us that $3n + 2$ is even, which is a contradiction of our assumption that $3n + 2$ is odd.
- Therefore, we have shown that if $3n + 2$ is odd, then $n$ is also odd.

**Note:** If we prove this by contraposition, we use a lot of the same mechanics/algebra!
- But the line of argumentation is different.

We can also use proof by contradiction in cases where were the theorem to be proved is not of the form $p \rightarrow q$. 
**Prove:** At least 10 of any 64 dates fall on the same day of the week

**Proof:**

- Let \( p \equiv \) “At least 10 of any 64 dates fall on the same day of the week”
- Assume \( \neg p \) is true, that is “64 dates can be chosen such that at most 9 fall on the same day of the week”
- When choosing dates, since there are 7 days in a week, at most \( 7 \times 9 = 63 \) dates can be chosen in total
- This is a contradiction of the statement that we can chose 64 dates
- Therefore, the assumption is false, and we can conclude that at least 10 of any 64 dates fall on the same day of the week.  

\( \Box \)

This proof is an example of the **pigeonhole principle**, which we will study during our combinatorics unit.
Problem 1: Prove the following claims

a) Use a direct proof to show that the square of an even number is an even number.
   ➢ Note that we proved the converse earlier, so this will prove it is an “if and only if” relationship

b) Use proof by contraposition to show that if \( n \) is an integer and \( n^3 + 5 \) is odd, then \( n \) is even.

Problem 2: On Top Hat
Sadly, not all theorems are of the form $p \rightarrow q$

Sometimes, we need to prove a theorem of the form:

$$p_1 \lor p_2 \lor ... \lor p_n \rightarrow q$$

**Note:** $p_1 \lor p_2 \lor ... \lor p_n \rightarrow q$

So, we might need to examine *multiple cases*!
Prove that $n^2 + 1 \geq 2n$ where $n$ is a positive integer with $1 \leq n \leq 4$

Proof:

- $n = 1$:
  
- $n = 2$:
  
- $n = 3$:
  
- $n = 4$:
  
Since we have verified each case, we have shown that $n^2 + 1 \geq 2n$ where $n$ is a positive integer with $1 \leq n \leq 4$. ☐

With only 4 cases to consider, exhaustive proof was a good choice!
Sometimes, exhaustive proof isn’t an option, but we still need to examine multiple possibilities.

**Example:** Prove the triangle inequality. That is, if $x$ and $y$ are real numbers, then $|x| + |y| \geq |x + y|$.

Clearly, we can’t use exhaustive proof here since there are infinitely many real numbers to consider.

We also can’t use a simple direct proof either, since our proof depends on the signs of $x$ and $y$.

**What should we do?**
**Example:** Prove that if $x$ and $y$ are real numbers, then $|x| + |y| \geq |x + y|$.

- **Note:** If $x \geq 0$, $|x| = x$, otherwise $|x| = -x$

**Cases:**

1) $x \geq 0$ and $y \geq 0$
   - $|x| + |y| = x + y$ and $|x + y| = x + y$
   - $x + y \geq x + y$ ✓

2) $x < 0$ and $y < 0$
   - $|x| + |y| = -x - y$ and $|x + y| = -x - y$
   - $-x - y \geq -x - y$ ✓

3) $x \geq 0$ and $y < 0$
   - If $x \geq |y|$, then $|x + y| = x - |y|$ and $|x| + |y| = x + |y|$
     - $x + |y| \geq x - |y|$ ✓
   - If $x < |y|$, then $|x + y| = |y| - x$ and $|x| + |y| = x + |y|$
     - $|y| + x \geq |y| - x$ ✓

4) Symmetrical to Case 3 □
Mistake 1: Proof by “a few cases” is not equivalent to proof by cases.

This is a “there exists” proof, not a “for all” proof!

Example: Prove that all odd numbers are prime.

“Proof:”

- Case (i): The number 1 is both odd and prime
- Case (ii): The number 3 is both odd and prime
- Case (iii): The number 5 is both odd and prime
- Case (iv): The number 7 is both odd and prime

Thus, we have shown that odd numbers are prime. □
Making mistakes when using proof by cases is all too easy!

_Mistake 2:_ Leaving out critical cases.

**Example:** Prove that $x^2 > 0$ for all integers $x$

**“Proof:”**
- Case (i): Assume that $x < 0$. Since the product of two negative numbers is always positive, $x^2 > 0$.
- Case (ii): Assume that $x > 0$. Since the product of two positive numbers is always positive, $x^2 > 0$.

Since we have proven the claim for all cases, we can conclude that $x^2 > 0$ for all integers $x$. □

*What about the case in which $x = 0$?*
Sometimes we need to prove the existence of a given element

There are two ways to do this

The constructive approach

The non-constructive approach
A constructive existence proof

**Prove:** Show that there is a positive integer that can be written as the sum of cubes of positive integers in two different ways.

**Proof:**

Obviously, the claim has been proven because we have shown that a specific instance of the claim is valid.

*Constructive existence proofs are really just instances of “existential generalization.”*
A non-constructive existence proof

Prove: Show that there exist two irrational numbers $x$ and $y$ such that $x^y$ is rational.

Proof:

Note: We don’t know whether $\sqrt{2}^{\sqrt{2}}$ is rational or irrational. However, in either case, we can use it to construct a rational number.
The scientific process is not always straightforward...

Conjecture → Gather evidence, prove lemmas → Prove theorem
Proof strategies can help preserve your sanity

Proof strategies help us...

Organize our problem solving approach

Effectively use all of the tools at our disposal

Develop a coherent plan of attack
Today we’ll discuss four types of strategy:

1. Forward reasoning
2. Backward reasoning
3. Searching for counterexamples
4. Adapting existing proofs
Sometimes forward reasoning doesn’t work

In these cases, it is often helpful to reason backwards, starting with the goal that we want to prove.

**Example:** Prove that given two distinct positive real numbers x and y, the arithmetic mean of x and y is always greater than the geometric mean of x and y.

**Sanity check:** Let x=8 and y=4. \((8+4)/2 = 6\). \(\sqrt{8 \times 4} = \sqrt{32} \approx 5.66.\) 6 > 5.66 ✔
Prove that \((x+y)/2 > \sqrt{xy}\) for all distinct pairs of positive real numbers \(x\) and \(y\).

**Proof:**

Since \((x - y)^2 > 0\) whenever \(x \neq y\), the final inequality is true. Since all of these inequalities are equivalent, it follows that \((x + y)/2 > \sqrt{xy}\). □
Other times, searching for a counterexample is helpful

Proof by counterexample is helpful if:

- Proof attempts repeatedly fail
- The conjecture to be proven looks “funny”

**Example:** Prove that every positive integer is the sum of two squares.

This seems suspicious to me, since other factorizations (e.g., prime factorizations) can be complex.

**Counterexample:**
3 is not the sum of two squares, so the claim is false. ☐
These four proof strategies are just a start!

A great tool for programmers AND logicians!

When trying to prove a new conjecture, a good “meta strategy” is to:

1. If possible, try to reuse an existing proof (analogy!)
2. If the conjecture looks fishy, check for a counterexample
3. Attempt a “real” proof
   a) Apply the forward reasoning strategy
   b) Or, apply the backward reasoning strategy
   c) Possibly alternate between forward and backward reasoning

Unfortunately, not every proof can be solved using this nice little meta strategy...

In fact, there are many, many proof strategies out there, and NONE of them can be guaranteed to find a proof!
In-class exercises

Problem 2:  Prove that there exists a positive integer that is equal to the sum of all positive integers less than it. Is your proof constructive or non-constructive?

Problem 3:  Prove that there is no positive integer \( n \) such that \( n^2 + n^3 = 100 \).

Top Hat
Final Thoughts

- Proving theorems is not always straightforward

- Having several proof strategies at your disposal will make a huge difference in your success rate!

- We are “done” with our intro to logic and proofs

- Next lecture:
  - Intro to set theory
  - Please read sections 2.1 and 2.2