Discrete Structures for Computer Science

William Garrison
bill@cs.pitt.edu
6311 Sennott Square

Lecture #3: Logic Puzzles and Propositional Equivalence

Based on materials developed by Dr. Adam Lee
Today’s topics

- Logic puzzles
- Propositional equivalences
Alice and Bob are technical support agents. If an agent is having a bad day, they will always lie to you. If an agent is having a good day, they will always tell you the truth. Alice tells you that Bob is having a bad day. Bob tells you that he and Alice are both having the same type of day. Can you trust the advice you receive from Alice during your call?

How do we solve this type of problem?
Solving logic puzzles is easy!

Step 1: Identify rules and constraints

Step 2: Assign propositions to key concepts

Step 3: Make assumptions and reason logically!
Alice and Bob are technical support agents. If an agent is having a bad day, they will always lie to you. If an agent is having a good day, they will always tell you the truth. Alice tells you that Bob is having a bad day. Bob tells you that he and Alice are both having the same type of day. Can you trust the advice you receive from Alice during your call?

Step 1: Identify the rules of the puzzle

Step 2: Assign propositions to the key concepts in the puzzle
Step 3: Make assumptions and reason logically
Another example

Consider a group of friends: Frank, Anna, and Chris. If Frank is not the oldest, then Anna is. If Anna is not the youngest, then Chris is the oldest. Determine the relative ages of Frank, Anna, and Chris.

Propositions:

Rules:
Step 3: Make assumptions and reason logically
Sometimes no solution is a solution!

Alice and Bob are technical support agents. Alice says “I am having a good day.” Bob says “I am having a good day.” Can you trust either Alice or Bob?

**Step 1:** Identify rules

**Step 2:** Assign propositions
Step 3: Make assumptions and reason logically
In-class exercises

Problem 1: Alice and Bob are technical support agents working to fix your computer. Alice tells you that Bob is having a bad day today and that you should expect a long wait before your computer is fixed. Bob tells you not to worry, Alice is just having a bad day—your computer will be ready in no time.

Question: Can you draw any conclusions about when your computer will be fixed? If so, what can you learn?
Propositional equivalences: preliminaries

Definition: A tautology is a compound proposition that is always true, regardless of the truth values of the propositions occurring within it.

Definition: A contradiction is a compound proposition that is always false, regardless of the truth values of the propositions occurring within it.

Definition: A contingency is a compound proposition whose truth value is dependent on the propositions occurring within it.
Examples

Are the following compound propositions tautologies, contradictions, or contingencies?

- \( p \lor \neg p \) tautology
- \( \neg p \land p \) contradiction
- \( p \lor q \) contingency
What are logical equivalences and why are they useful?

**Definition:** Compound propositions $p$ and $q$ are **logically equivalent** if $p \leftrightarrow q$ is a tautology. The notation $p \equiv q$ means that $p$ and $q$ are logically equivalent.

Logical equivalences are extremely useful!
- Aid in the construction of proofs
- Allow us to simplify compound propositions

**Example:** $p \rightarrow q \equiv \neg p \lor q$

How do we prove this type of statement?
It is easy to prove propositional equivalences.

We can prove simple logical equivalences using our good friend the truth table!

**Prove:** $p \rightarrow q \equiv \neg p \lor q$

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$\neg p$</th>
<th>$\neg p \lor q$</th>
<th>$p \rightarrow q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td></td>
<td></td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td></td>
<td></td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td></td>
<td></td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td></td>
<td></td>
<td>F</td>
</tr>
</tbody>
</table>
DeMorgan’s laws allow us to distribute negation over compound propositions

Two laws:

- \( \neg(p \lor q) \equiv \neg p \land \neg q \)
- \( \neg(p \land q) \equiv \neg p \lor \neg q \)

Prove: \( \neg(p \lor q) \equiv \neg p \land \neg q \)

If “p or q” isn’t true, then neither p nor q is true

If “p and q” isn’t true, then at least one of p or q is false

<table>
<thead>
<tr>
<th>p</th>
<th>q</th>
<th>p \lor q</th>
<th>\neg(p \lor q)</th>
<th>\neg p</th>
<th>\neg q</th>
<th>\neg p \land \neg q</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
</tbody>
</table>
Use DeMorgan’s laws to negate the following expressions:

- “Bob is wearing blue pants and a sweatshirt”
  - \( b \land s \)
  - \( \neg (b \land s) \equiv \neg b \lor \neg s \)
  - Bob is not wearing blue pants or is not wearing a sweatshirt

- “I will drive or I will walk”
  - \( d \lor w \)
  - \( \neg (d \lor w) \equiv \neg d \land \neg w \)
  - I will not drive and I will not walk
In-class exercises

Problem 2: Prove that \( \neg(p \land q) \) and \( \neg p \lor \neg q \) are logically equivalent, i.e., \( \neg(p \land q) \equiv \neg p \lor \neg q \). This is the second DeMorgan’s law.

Problem 3: Use DeMorgan’s laws to negate the following propositions:

- Today I will go running or ride my bike
- Tom likes both pizza and beer

One more on Top Hat
Sometimes using truth tables to prove logical equivalencies can become cumbersome.

Recall that for an equivalence with $n$ propositions, we need to build a truth table with $2^n$ rows:

- Fine for tables with $n = 2$, 3, or 4
- Consider $n = 30$—we would need 1,073,741,824 rows in the truth table!

Another option: Direct manipulation of compound propositions using known logical equivalencies.
There are many useful logical equivalences

<table>
<thead>
<tr>
<th>Equivalence</th>
<th>Name</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>
More useful logical equivalences

<table>
<thead>
<tr>
<th>Equivalence</th>
<th>Name</th>
</tr>
</thead>
<tbody>
<tr>
<td>(p \lor q) \land r</td>
<td>\land (p \lor (q \land r))</td>
</tr>
<tr>
<td>(p \land q) \lor r</td>
<td>\lor (p \land (q \lor r))</td>
</tr>
<tr>
<td>(p \lor q) \land (p \lor r)</td>
<td>(p \lor r) \land (p \lor q)</td>
</tr>
<tr>
<td>(p \land q) \lor (p \land r)</td>
<td>(p \land r) \lor (p \land q)</td>
</tr>
</tbody>
</table>

**More equivalencies in the book!**
Prove that \((p \land q) \rightarrow (p \lor q)\) is a tautology
Prove: \((p \rightarrow q) \lor (p \rightarrow r) \equiv p \rightarrow (q \lor r)\)
Final Thoughts

- Logic can help us solve real world problems and play challenging games

- Logical equivalences help us simplify complex propositions and construct proofs
  - More on proofs later in the course

- Next:
  - Predicate logic and quantification
  - Please read section 1.4