Discrete Structures for Computer Science

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Lecture #2: Propositional Logic

Based on materials developed by Dr. Adam Lee
Today’s Topic: Propositional Logic

- What is a proposition?
- Logical connectives and truth tables
- Translating between English and propositional logic
Logic is the basis of all mathematical and analytical reasoning.

Given a collection of known truths, logic allows us to deduce new truths.

**Example**

**Base facts:**
- If it is raining, I will not go outside
- If I am inside, Lisa will stay home
- Lisa and I always play video games if we are together during the weekend
- Today is a rainy Saturday

**Conclusion:** Lisa and I will play video games today

Logic allows us to advance mathematics through an iterative process of *conjecture* and *proof*.
Propositional logic is a very simple logic

**Definition:** A proposition is a precise statement that is either **true** or **false**, but not both.

**Examples:**
- $2 + 2 = 4$ (**true**)
- All dogs have 3 legs (**false**)
- $x^2 < 0$ (**false**)
- Washington, D.C. is the capital of the USA (**true**)
Not all statements are propositions

- Eliana is cool
  - “Cool” is a subjective term.

- \( x^3 < 0 \)
  - True if \( x < 0 \), false otherwise.

- Springfield is the capital
  - True in Illinois, false in Massachusetts.
We can use logical connectives to build complex propositions

We will discuss the following logical connectives:

- ¬ (not)
- ∧ (conjunction / and)
- ∨ (disjunction / or)
- ⊕ (exclusive disjunction / xor)
- → (implication)
- ↔ (biconditional)
The negation of a proposition is true iff the proposition is false.

The truth table for negation:

<table>
<thead>
<tr>
<th>What we know</th>
<th>What we want to know</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p$</td>
<td>$\neg p$</td>
</tr>
</tbody>
</table>

One row for each possible value of “what we know”.

(I’ll sometimes use $\top$ and $\bot$)
Negation Examples

Negate the following propositions

- Today is Monday
- $21 \times 2 = 42$

What is the truth value of the following propositions

- $\neg(9$ is a prime number$)$
- $\neg$(Pittsburgh is in Pennsylvania)
The conjunction of two propositions is true iff both propositions are true

The truth table for conjunction:

<table>
<thead>
<tr>
<th>p</th>
<th>q</th>
<th>p \land q</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
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\(2^2 = 4\) rows since we know both \(p\) and \(q\)!
Disjunction

The **disjunction** of two propositions is true iff *at least one* proposition is true

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<tbody>
<tr>
<td><strong>p</strong></td>
<td><strong>q</strong></td>
<td><strong>p ∨ q</strong></td>
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</table>

The truth table for disjunction
Conjunction and disjunction examples

Let:

- $p \equiv x^2 \geq 0$
- $q \equiv \text{A lion weighs less than a mouse}$
- $r \equiv 10 < 7$
- $s \equiv \text{Pittsburgh is located in Pennsylvania}$

What are the truth values of these expressions:

- $p \land q$
- $p \land s$
- $p \lor q$
- $q \lor r$
Problem 1: Let \( p \equiv 2+2=5, \ q \equiv \text{eagles can fly}, \ r \equiv 1=1. \) Determine the value for each of the following:

- \( p \land q \)
- \( \neg p \lor q \)
- \( p \lor (q \land r) \)
- \( p \lor (q \land r) \)
- \( (p \lor q) \land (\neg r \lor \neg p) \)
The exclusive or of two propositions is true iff exactly one proposition is true

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<th>p</th>
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<th>p ⊕ q</th>
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The truth table for exclusive or

Note: Exclusive or is typically used to natural language to identify choices. For example, “You may have a soup or salad with your entree.”
The implication $p \rightarrow q$ is false if $p$ is true, and $q$ is false; $p \rightarrow q$ is true otherwise.

Terminology
- $p$ is called the hypothesis
- $q$ is called the conclusion

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<td>$p \rightarrow q$</td>
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The truth table for implication
The implication $p \rightarrow q$ can be read in a number of (equivalent) ways:

- If $p$ then $q$
- $p$ only if $q$
- $p$ is sufficient for $q$
- $q$ whenever $p$
Implication examples

Let:

- $p \equiv$ Jane gets a 100% on her final exam
- $q \equiv$ Jane gets an A on her final exam

What are the truth values of these implications:

- $p \rightarrow q$
- $q \rightarrow p$
Given an implication $p \rightarrow q$:

- $q \rightarrow p$ is its converse
- $\neg q \rightarrow \neg p$ is its contrapositive
- $\neg p \rightarrow \neg q$ is its inverse

**Note:** An implication and its contrapositive always have the same truth value.
The biconditional \( p \leftrightarrow q \) is true if and only if \( p \) and \( q \) assume the same truth value.

The truth table for the biconditional:

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<th>p ( \leftrightarrow ) q</th>
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The truth table for the biconditional.

**Note:** The biconditional statement \( p \leftrightarrow q \) is often read as “\( p \) if and only if \( q \)” or “\( p \) is a necessary and sufficient condition for \( q \)”.
Truth tables can also be made for more complex expressions.

**Example:** What is the truth table for \((p \land q) \rightarrow \neg r\)?

<table>
<thead>
<tr>
<th>p</th>
<th>q</th>
<th>r</th>
<th>(p \land q) \rightarrow \neg r</th>
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</table>
Like mathematical operators, logical operators are assigned precedence levels

1. Negation
   - $\neg q \lor r$ means $(\neg q) \lor r$, not $\neg(q \lor r)$

2. Conjunction

3. Disjunction
   - $q \land r \lor s$ means $(q \land r) \lor s$, not $q \land (r \lor s)$

4. Implication
   - $q \land r \rightarrow s$ means $(q \land r) \rightarrow s$, not $q \land (r \rightarrow s)$

5. Biconditional

In general, we will try to use parenthesis to disambiguate these types of expressions.
Problem 2: Show that an implication \( p \rightarrow q \) and its contrapositive \( \neg q \rightarrow \neg p \) always have the same value

- **Hint:** Construct two truth tables

Problem 3: Construct the truth table for the compound proposition \( p \land (\neg q \lor r) \rightarrow s \)
English sentences can often be translated into propositional sentences

But *why* would we do that?

Philosophy and epistemology

Reasoning about law

Verifying complex system specifications
Example #1

Example: You can see an R-rated movie only if you are over 17 or you are accompanied by your legal guardian.

Let:

Find logical connectives

Translate fragments

Create logical expression
Example #2

**Example:** You can have free coffee if you are a senior citizen and it is a Tuesday

Let:
Example #3

Example: If you are under 17 and are not accompanied by your legal guardian, then you cannot see the R-rated movie.

Let:

Note: The above translation is the contrapositive of the translation from example 1!
Logic also helps us understand bitwise operations

- Computers represent data as sequences of bits
  - e.g., 0101 1101 1010 1111

- Bitwise logical operations are often used to manipulate these data

- If we treat 1 as **true** and 0 as **false**, our logic truth tables tell us how to carry out bitwise logical operations
Bitwise logic examples

\[
\begin{array}{c}
1010 \ 1110 \\
\wedge \ 1110 \ 1010 \\
\hline
1110 \ 1010
\end{array}
\quad
\begin{array}{c}
1010 \ 1110 \\
\lor \ 1110 \ 1010 \\
\hline
1110 \ 1010
\end{array}
\]

\[
\begin{array}{c}
1010 \ 1110 \\
\oplus \ 1110 \ 1010 \\
\hline
1110 \ 1010
\end{array}
\]
In-class Exercises

Problem 4: Translate the following sentences

- On Top Hat

Problem 5: Solve the following bitwise problems

\[
\begin{array}{l}
1011\ 1000 \\
\oplus\ 1010\ 0110 \\
\hline
1010\ 0110
\end{array}
\]

\[
\begin{array}{l}
1011\ 1000 \\
\land\ 1010\ 0110 \\
\hline
1010\ 0110
\end{array}
\]
Final Thoughts

- Propositional logic is a simple logic that allows us to reason about a variety of concepts

- In recitation:
  - More examples and practice problems
  - Be sure to attend!

- Next:
  - Logic puzzles and logical equivalences
  - Please read sections 1.2 and 1.3

  ➢ In general: do the assigned reading!