Discrete Structures for Computer Science

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Lecture #2: Propositional Logic
Today’s Topic: Propositional Logic

- What is a proposition?
- Logical connectives and truth tables
- Translating between English and propositional logic
Logic is the basis of all mathematical and analytical reasoning

Given a collection of known truths, logic allows us to deduce new truths

**Example**

Base facts:
- If it is raining, I will not go outside
- If I am inside, Lisa will stay home
- Lisa and I always play video games if we are together during the weekend
- Today is a rainy Saturday

**Conclusion:** Lisa and I will play video games today

Logic allows us to advance mathematics through an iterative process of **conjecture** and **proof**
Propositional logic is a very simple logic

Definition: A proposition is a precise statement that is either true or false, but not both.

Examples:
- $2 + 2 = 4$ (true)
- All dogs have 3 legs (false)
- $x^2 < 0$ (false)
- Washington, D.C. is the capital of the USA (true)
Not all statements are propositions

- Charlie is handsome
  - “Handsome” is a subjective term.

- $x^3 < 0$
  - True if $x < 0$, false otherwise.

- Springfield is the capital
  - True in Illinois, false in Massachusetts.
We can use logical connectives to build complex propositions.

We will discuss the following logical connectives:

- \( \neg \) (not)
- \( \land \) (conjunction / and)
- \( \lor \) (disjunction / or)
- \( \oplus \) (exclusive disjunction / xor)
- \( \rightarrow \) (implication)
- \( \leftrightarrow \) (biconditional)
The negation of a proposition is **true** iff the proposition is **false**.

<table>
<thead>
<tr>
<th>What we know</th>
<th>What we want to know</th>
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<tbody>
<tr>
<td>p</td>
<td>( \neg p )</td>
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The truth table for negation

One row for each possible value of "what we know"
Negate the following propositions

- Today is Monday
- \(21 \times 2 = 42\)

What is the truth value of the following propositions

- \(\neg(9 \text{ is a prime number})\)
- \(\neg(\text{Pittsburgh is in Pennsylvania})\)
Conjunction

The **conjunction** of two propositions is true iff both propositions are true

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<thead>
<tr>
<th>p</th>
<th>q</th>
<th>$p \land q$</th>
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The truth table for conjunction

$2^2 = 4$ rows since we know both $p$ and $q$!
The **disjunction** of two propositions is true if *at least one* proposition is true.

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The truth table for disjunction.
Conjunction and disjunction examples

Let:

- \( p \equiv x^2 \geq 0 \)
- \( q \equiv \text{A lion weighs less than a mouse} \)
- \( r \equiv 10 < 7 \)
- \( s \equiv \text{Pittsburgh is located in Pennsylvania} \)

What are the truth values of these expressions:

- \( p \land q \)
- \( p \land s \)
- \( p \land s \)
- \( p \lor q \)
- \( q \lor r \)
Problem 1: Let $p \equiv 2+2=5$, $q \equiv$ eagles can fly, $r \equiv 1=1$. Determine the value for each of the following:

- $p \land q$
- $\neg p \lor q$
- $p \lor (q \land r)$
- $p \lor (q \land r)$
- $(p \lor q) \land (\neg r \lor \neg p)$
The exclusive or of two propositions is true if exactly one proposition is true.

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The truth table for exclusive or.

Note: Exclusive or is typically used to natural language to identify choices. For example “You may have a soup or salad with your entree.”
The implication $p \rightarrow q$ is **false** if $p$ is **true** and $q$ is **false**, and **true** otherwise.

**Terminology**
- $p$ is called the hypothesis
- $q$ is called the conclusion

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The truth table for implication
The implication $p \rightarrow q$ can be read in a number of (equivalent) ways:

- If $p$ then $q$
- $p$ only if $q$
- $p$ is sufficient for $q$
- $q$ whenever $p$
Implication examples

Let:

- \( p \equiv \) Jane gets a 100% on her final
- \( q \equiv \) Jane gets an A

What are the truth values of these implications:

- \( p \rightarrow q \)
- \( q \rightarrow p \)
Other conditional statements

Given an implication $p \rightarrow q$:

- $q \rightarrow p$ is its converse
- $\neg q \rightarrow \neg p$ is its contrapositive
- $\neg p \rightarrow \neg q$ is its inverse

Note: An implication and its contrapositive always have the same truth value

Why might this be useful?
The biconditional $p \leftrightarrow q$ is true if and only if $p$ and $q$ assume the same truth value.

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The truth table for the biconditional

**Note:** The biconditional statement $p \leftrightarrow q$ is often read as “$p$ if and only if $q$” or “$p$ is a necessary and sufficient condition for $q$.”
Truth tables can also be made for more complex expressions.

**Example:** What is the truth table for \((p \land q) \rightarrow \neg r\)?

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2^3 = 8 rows

Subexpressions of “what we want to know”

What we want to know
Like mathematical operators, logical operators are assigned precedence levels.

1. Negation
   - $\neg q \lor r$ means $(\neg q) \lor r$, not $\neg (q \lor r)$

2. Conjunction

3. Disjunction
   - $q \land r \lor s$ means $(q \land r) \lor s$, not $q \land (r \lor s)$

4. Implication
   - $q \land r \rightarrow s$ means $(q \land r) \rightarrow s$, not $q \land (r \rightarrow s)$

5. Biconditional

In general, we will try to use parenthesis to disambiguate these types of expressions.
In-class Exercises

Problem 2: Show that an implication $p \rightarrow q$ and its contrapositive $\neg q \rightarrow \neg p$ always have the same value

**Hint:** Construct two truth tables

Problem 3: Construct the truth table for the compound proposition $p \land (\neg q \lor r) \rightarrow s$
English sentences can often be translated into propositional sentences

But *why* would we do that?

Philosophy and epistemology

Reasoning about law

Verifying complex system specifications
Example #1

Example: You can see an R-rated movie only if you are over 17 or you are accompanied by your legal guardian.

Let:

Find logical connectives

Translate fragments

Create logical expression
Example #2

**Example:** You can have free coffee if you are a senior citizen and it is a Tuesday.

**Let:**
Example #3

Example: If you are under 17 and are not accompanied by your legal guardian, then you cannot see the R-rated movie.

Let:

Note: The above translation is the contrapositive of the translation from example 1!
Logic also helps us understand bitwise operations

- Computers represent data as sequences of bits
  - e.g., 0101 1101 1010 1111

- Bitwise logical operations are often used to manipulate these data

- If we treat 1 as true and 0 as false, our logic truth tables tell us how to carry out bitwise logical operations
Bitwise logic examples

\[ \begin{array}{c}
\land & 1010 & 1110 \\
& 1110 & 1010 \\
\hline
& 1110 & 1010 \\
\end{array} \quad \begin{array}{c}
\lor & 1010 & 1110 \\
& 1110 & 1010 \\
\hline
& 1110 & 1010 \\
\end{array} \quad \begin{array}{c}
\oplus & 1010 & 1110 \\
& 1110 & 1010 \\
\hline
& 1110 & 1010 \\
\end{array} \]
In-class Exercises

Problem 4: Translate the following sentences

- If it is raining and today is Saturday then I will either play video games or watch a movie.
- You get a free salad only if you order off of the extended menu and it is a Wednesday.

Problem 5: Solve the following bitwise problems

\[\begin{align*}
1011 & \ 1000 \\
\oplus 1010 & \ 0110 \\
\hline
1010 & \ 0110
\end{align*}\] 

\[\begin{align*}
1011 & \ 1000 \\
\wedge 1010 & \ 0110 \\
\hline
1010 & \ 0110
\end{align*}\]
Final Thoughts

- Propositional logic is a simple logic that allows us to reason about a variety of concepts.

- In recitation:
  - More examples and practice problems
  - Be sure to attend!

- Next lecture:
  - Logic puzzles and logical equivalences
  - Please read sections 1.2 and 1.3
    - In general: do the assigned reading!