Discrete Structures for Computer Science

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Lecture #2: Propositional Logic
Today’s Topic: Propositional Logic

- What is a proposition?
- Logical connectives and truth tables
- Translating between English and propositional logic
Logic is the basis of all mathematical and analytical reasoning.

Given a collection of known truths, logic allows us to deduce new truths.

**Example**

**Base facts:**
- If it is raining, I will not go outside.
- If I am inside, Lisa will stay home.
- Lisa and I always play video games if we are together during the weekend.
- Today is a rainy Saturday.

**Conclusion:** Lisa and I will play video games today.

Logic allows us to advance mathematics through an iterative process of **conjecture** and **proof**.
Definition: A proposition is a precise statement that is either true or false, but not both.

Examples:

- $2 + 2 = 4$ (true)
- All dogs have 3 legs (false)
- $x^2 < 0$ (false)
- Washington, D.C. is the capital of the USA (true)
Not all statements are propositions

- Charlie is handsome
  - “Handsome” is a subjective term.

- $x^3 < 0$
  - True if $x < 0$, false otherwise.

- Springfield is the capital
  - True in Illinois, false in Massachusetts.
We can use logical connectives to build complex propositions

We will discuss the following logical connectives:

- \( \neg \) (not)
- \( \land \) (conjunction / and)
- \( \lor \) (disjunction / or)
- \( \oplus \) (exclusive disjunction / xor)
- \( \rightarrow \) (implication)
- \( \leftrightarrow \) (biconditional)
The **negation** of a proposition is **true** iff the proposition is **false**

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<thead>
<tr>
<th>What we know</th>
<th>What we want to know</th>
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<tr>
<td>p</td>
<td>¬p</td>
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The truth table for negation

(I’ll sometimes use ⊤ and ⊥)
Negation Examples

Negate the following propositions

- Today is Monday
  - Today is not Monday

- 21 * 2 = 42
  - 21 * 2 ≠ 42

What is the truth value of the following propositions

- ¬(9 is a prime number)
  - true

- ¬(Pittsburgh is in Pennsylvania)
  - false
The **conjunction** of two propositions is true iff both propositions are true.

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<tr>
<th>p</th>
<th>q</th>
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The truth table for conjunction

$2^2 = 4$ rows since we know both $p$ and $q$!
Disjunction

The disjunction of two propositions is true iff at least one proposition is true

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<th>p</th>
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The truth table for disjunction
Conjunction and disjunction examples

Let:
- \( p \equiv x^2 \geq 0 \)
- \( q \equiv \text{A lion weighs less than a mouse} \)
- \( r \equiv 10 < 7 \)
- \( s \equiv \text{Pittsburgh is located in Pennsylvania} \)

What are the truth values of these expressions:
- \( p \land q \)
- \( p \land s \)
- \( p \lor q \)
- \( q \lor r \)
Problem 1: Let $p \equiv 2+2=5$, $q \equiv$ eagles can fly, $r \equiv 1=1$. Determine the value for each of the following:

- $p \land q$
- $\neg p \lor q$
- $p \lor (q \land r)$
- $p \lor (q \land r)$
- $(p \lor q) \land (\neg r \lor \neg p)$
The exclusive or of two propositions is true iff exactly one proposition is true

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The truth table for exclusive or

Note: Exclusive or is typically used to natural language to identify choices. For example “You may have a soup or salad with your entree.”
Implication

The implication $p \rightarrow q$ is false if $p$ is true and $q$ is false, and true otherwise.

Terminology
- $p$ is called the hypothesis
- $q$ is called the conclusion

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The truth table for implication
The implication $p \rightarrow q$ can be read in a number of (equivalent) ways:

- If $p$ then $q$
- $p$ only if $q$
- $p$ is sufficient for $q$
- $q$ whenever $p$
Implication examples

Let:

- $p \equiv$ Jane gets a 100% on her final exam
- $q \equiv$ Jane gets an A on her final exam

What are the truth values of these implications:

- $p \rightarrow q$
- $q \rightarrow p$
Other conditional statements

Given an implication $p \rightarrow q$:

- $q \rightarrow p$ is its converse
- $\neg q \rightarrow \neg p$ is its contrapositive
- $\neg p \rightarrow \neg q$ is its inverse

Note: An implication and its contrapositive always have the same truth value
The biconditional $p \iff q$ is true if and only if $p$ and $q$ assume the same truth value.

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The truth table for the biconditional.

Note: The biconditional statement $p \iff q$ is often read as “$p$ if and only if $q$” or “$p$ is a necessary and sufficient condition for $q$. “
Truth tables can also be made for more complex expressions.

**Example:** What is the truth table for \((p \land q) \rightarrow \neg r\)?

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Like mathematical operators, logical operators are assigned precedence levels

1. Negation
   - \( \neg q \lor r \) means \((\neg q) \lor r\), not \(\neg(q \lor r)\)

2. Conjunction
3. Disjunction
   - \( q \land r \lor s \) means \((q \land r) \lor s\), not \(q \land (r \lor s)\)

4. Implication
   - \( q \land r \rightarrow s \) means \((q \land r) \rightarrow s\), not \(q \land (r \rightarrow s)\)

5. Biconditional

In general, we will try to use parenthesis to disambiguate these types of expressions.
Problem 2: Show that an implication $p \rightarrow q$ and its contrapositive $\neg q \rightarrow \neg p$ always have the same value

- **Hint:** Construct two truth tables

Problem 3: Construct the truth table for the compound proposition $p \land (\neg q \lor r) \rightarrow s$
English sentences can often be translated into propositional sentences.

But *why* would we do that?

Philosophy and epistemology

Reasoning about law

Verifying complex system specifications
Example: You can see an R-rated movie only if you are over 17 or you are accompanied by your legal guardian.

Let:

Find logical connectives

Translate fragments

Create logical expression
Example #2

**Example**: You can have free coffee if you are a senior citizen and it is a Tuesday

Let:
Example: If you are under 17 and are not accompanied by your legal guardian, then you cannot see the R-rated movie.

Let:

Note: The above translation is the contrapositive of the translation from example 1!
Logic also helps us understand bitwise operations

- Computers represent data as sequences of bits
  - e.g., 0101 1101 1010 1111

- Bitwise logical operations are often used to manipulate these data

- If we treat 1 as **true** and 0 as **false**, our logic truth tables tell us how to carry out bitwise logical operations
Bitwise logic examples

\[ \land \begin{array}{c} 1010 \\ 1110 \\ 1010 \end{array} \begin{array}{c} 1110 \\ 1010 \end{array} \begin{array}{c} \land \\ 1110 \\ 1010 \end{array} \begin{array}{c} 1010 \\ 1110 \\ 1010 \end{array} \]

\[ \lor \begin{array}{c} 1010 \\ 1110 \\ 1010 \end{array} \begin{array}{c} 1010 \\ 1110 \\ 1010 \end{array} \begin{array}{c} \lor \\ 1110 \\ 1010 \end{array} \begin{array}{c} 1010 \\ 1110 \\ 1010 \end{array} \]

\[ \oplus \begin{array}{c} 1010 \\ 1110 \\ 1010 \end{array} \begin{array}{c} 1010 \\ 1110 \\ 1010 \end{array} \begin{array}{c} \oplus \\ 1110 \\ 1010 \end{array} \begin{array}{c} 1110 \\ 1010 \end{array} \]
Problem 4: Translate the following sentences
- If it is raining and today is Saturday then I will either play video games or watch a movie
- You get a free salad only if you order off of the extended menu and it is a Wednesday

Problem 5: Solve the following bitwise problems

\[ \begin{align*}
\oplus & \quad 1011 \ 1000 \\
& \quad 1010 \ 0110 \\
& \quad \underline{1010 \ 0110} \\
\wedge & \quad 1011 \ 1000 \\
& \quad 1010 \ 0110 \\
& \quad \underline{1010 \ 0110}
\end{align*} \]
Propositional logic is a simple logic that allows us to reason about a variety of concepts.

In recitation:
- More examples and practice problems
- Be sure to attend!

Next:
- Logic puzzles and logical equivalences
- Please read sections 1.2 and 1.3
  ➢ In general: do the assigned reading!