Instructions:

- This is a closed-book, closed-notes practice exam. It is recommended that you take it without using any outside resources, print or electronic.

- You may leave r-Permutations and r-Combinations written as $P(n,r)$ and $C(n,r)$, respectively, unless explicitly noted otherwise.
Problem 1: Recursion and induction

(a) Complete the recursive definition for \( d : \Sigma^* \rightarrow \Sigma^* \), where \( d(w) \) is a string where each of \( w \)'s characters is duplicated (e.g., \( d(\text{LEAF}) = \text{LLEEAAFF} \)).

Basis: Consider the empty string, \( \lambda \). Write the basis step of the definition of \( r \) for this case.

Recursive step: Consider a string that contains at least one character. Write the recursive step of the definition of \( r \) for this case.

(b) Let \( P(n) \equiv (E(X_1 + X_2 + \cdots + X_n) = E(X_1) + E(X_2) + \cdots + E(X_n)) \), where each \( X_i \) is a random variable drawn from sample space \( S \). Recall that we proved \( P(2) \) in lecture. Finish the proof that \( \forall k \geq 2P(k) \) by completing the inductive step, by proving (for an arbitrary \( k \)) that \( P(k) \rightarrow P(k + 1) \).
Problem 2: Counting

(a) Compute $P(8, 2)$. Show your work, and express your answer as an integer.

(b) Compute $C(10, 4)$. Show your work, and express your answer as an integer.

(c) Let $A = \{1, 2, \ldots, 21\}$. Let $B$ be an arbitrary subset of $A$ with cardinality 7 (i.e., $B \subseteq A$ and $|B| = 7$). Prove that there are 2 subsets of $B$ with the same sum.
Problem 3: The National Football League (NFL) contains 32 teams, 16 in the National Football Conference (NFC) and 16 in the American Football Conference (AFC). At the end of the NFL regular season, *seeding* ranks the 6 teams per conference that advance to the playoffs. Assume that seeding is decided based solely on the number of wins that a team has during the regular season, and that no two teams win the same number of games.

(a) How many unique ways can the 12 positions in the playoff brackets (6 in the NFC, 6 in the AFC) be seeded?

(b) How many ways are there to select the teams that do not make the playoffs?

(c) During a football game, officials called 8 offensive penalties. Each penalty was assessed either 5 yards, 10 yards, or 15 yards. Determine how many ways these 8 penalties could be assessed, if:

- the first 2 were for 10 yards and the last 1 was for 5 yards; OR
- the first 1 was for 10 yards and the last 2 were for 5 yards.
Problem 4: A standard deck of cards contains 52 cards. There are 13 different ranks (Ace, 2, 3, 4, 5, 6, 7, 8, 9, 10, Jack, Queen, King) and four suits (spades, clubs, hearts, diamonds). Each suit contains one card of each kind.

(a) A flush is a poker hand in which a player holds five cards of the same suit. Note that a straight flush or royal flush (i.e., hands in which the cards are of the same suit and in sequence) is not considered a flush. Mingzhi is being dealt a five-card poker hand. How many ways could she be dealt a flush?

(b) A four of a kind is a poker hand in which a player holds four cards of one rank, and one card of another rank (e.g., four 2s, one Ace). Andrew is being dealt a five-card poker hand. How many ways could he be dealt a four of a kind?

(c) A two pair is a poker hand in which a player holds two cards of one rank, two cards of a second rank, and one card of a third rank (e.g., two Jacks, two Queens, one 5). Tara is being dealt a five-card poker hand. How many ways could she be dealt a three of a kind?
Problem 5: Alice and Bob frequently play chess together, and Alice wins 70% of the time. They decide to have a best-of-5 chess series, where they will play until one of them has won 3 games. That is, they will play at least 3 games (in the case that one player wins the first 3 in a row) and at most 5 games (in the case that they are tied 2 games to 2 after the first 4 games). If player $p$ accumulates 3 wins after they play total of $n$ games, it is said that “$p$ wins in $n$ games”. Assume each game is independent.

(a) What is the probability that Alice sweeps the series (i.e., wins in 3 games)?

(b) What is the probability that Bob wins in 5 games?

(c) What is the probability that Alice will win in any number of games?
Problem 6: Consider two dice, a red die and a blue die. The red die is a fair six-sided die. The blue die is a biased ten-sided die. In particular: each prime face is equally likely to be rolled, each non-prime is equally likely to be rolled, and the probability of each prime face being rolled is twice that of each non-prime face. Recall that 1 is not prime.

(a) What is the probability distribution of the blue die?

(b) Let $B$ be a random variable that takes the value of the blue die when it is rolled once. What is the expected value of $B$?

(c) Let $R$ be a random variable that takes the value of the red die when it is rolled once. What is the expected value of $R$?

(d) If the red die and the blue die are both rolled, what is the expected value of the sum of the values that appear on the two dice?
Problem 7: Recall that Bayes’ theorem is defined as follows.

\[ p(F \mid E) = \frac{p(E \mid F)p(F)}{p(E \mid F)p(F) + p(E \mid F^C)p(F^C)} \]

Suppose that 60% of mobile apps are retired within one year of release. Among those, 5% achieve a million downloads. On the other hand, 20% of apps that are available for at least a year achieve a million downloads. Answer the following questions regarding the use of Bayes’ theorem (as written above) to determine the probability that an app will be retired within one year of release, given that it has achieved a million downloads.

(a) In English, what do each of the following probabilities represent?
   - \( p(F) \):
   - \( p(F^C) \):
   - \( p(E \mid F) \):
   - \( p(E \mid F^C) \):
   - \( p(F \mid E) \):

(b) What is the value of each of the following probabilities?
   - \( p(F) \):
   - \( p(F^C) \):
   - \( p(E \mid F) \):
   - \( p(E \mid F^C) \):

(c) Use Bayes’ theorem to calculate the probability that an app will be retired within one year of release, given that it has achieved a million downloads.