Max-flow, Min-cut

Network flow
Max-flow

- Maximize the total amount of flow from s to t subject to two constraints
  - Flow on edge e doesn’t exceed c(e)
  - For every node v ≠ {s, t}, incoming flow is equal to outgoing flow
Max-flow: Ford-Fulkerson

• Find paths from s to t using depth first search
• Find paths using the residual graph $G'$
Ford-Fulkerson: example

Depth first search

\[ f = \min(4, 10) = 4 \]
Ford-Fulkerson: DFS

\[
f = \min(10, 6) = 6
\]

\[
f = \min(7, 10, 7) = 7
\]
Ford-Fulkerson: Residual graph

$G$

$G'$

$\mathbf{s} \rightarrow \mathbf{t}$

$\mathbf{G'}$
Ford-Fulkerson: Residual graph

\[ s \rightarrow t \quad G \]

\[ s \rightarrow t \quad G' \]
Ford-Fulkerson: Residual graph

\[ G \]

\[ G' \]
Ford-Fulkerson: Residual graph

Keep work on for the rest of edges

Remove “0” edge (optional)
Ford-Fulkerson: Residual graph

Any more paths?

\[ f = \min(6,2,3,2,4) = 2 \]
Max flow = 6 + 7 +6 = 19
Edmonds-Karp

- Edmonds-Karp = Ford-Fulkerson + “Choose the augmenting path with the smallest number of edges” or “Choose the augmenting path with the largest bottle neck value”
Edmonds-Karp vs Ford-Fulkerson

Which one is the valid first choice of Edmonds-Karp?
Which one is the valid first choice of Ford-Fulkerson?
Min cut

- We want to remove some edges from the graph such that after removing the edges, there is no path from s to t
- The cost of removing e is equal to its capacity c(e)
- The minimum cut problem is to find a cut with minimum total cost
Min cut: approach

• “Subtract” the max-flow from the original graph
• Mark all nodes reachable from s. Call the set of reachable nodes A
• Now separate these nodes from the others
• Cut edges going from A to V − A
Min cut: example

G - G max flow = residual graph
Min cut: example

\[ G - \text{G max flow} = \text{residual graph} \]
Min cut: example

G – G max flow = residual graph
Min cut: example

G

G max flow

G – G max flow = residual graph
Min cut: example

G - G max flow = residual graph
Min cut: example

\[ G - G \text{ max flow} = \text{residual graph} \]
Min cut: example

G

G max flow

G – G max flow = residual graph
Min cut: example

$G - G \text{ max flow} = \text{residual graph}$
Min cut: example

\[ G - G \text{ max flow} = \text{residual graph} \]
Nodes reachable from $s$ (A)
Cut edges come from V - A

G – G max flow = residual graph

Cost of min cut = 4 + 7 + 2 + 6 = 19 = max flow value