Course administration

- **Homework 6 is out**
  Due on Friday, March 3, 2006 or earlier (TA office)

- **Homework 7 is out, due on March 17, 2006**

Course web page:
http://www.cs.pitt.edu/~milos/courses/cs441/
Proofs

**Basic proof methods:**
- Direct, Indirect, Contradiction, By Cases, Equivalences

**Proof of quantified statements:**
- **There exists x with some property P(x).**
  - It is sufficient to find one element for which the property holds.
- **For all x some property P(x) holds.**
  - Proofs of ‘For all x some property P(x) holds’ must cover all x and can be harder.
- **Mathematical induction** is the technique that can be applied to prove the universal statements for sets of positive integers or their associated sequences.

Mathematical induction

- Used to prove statements of the form $\forall x P(x)$ where $x \in \mathbb{Z}^+$

**Mathematical induction proofs** consists of two steps:
1) **Basis:** The proposition $P(1)$ is true.
2) **Inductive Step:** The implication $P(n) \rightarrow P(n+1)$, is true for all positive $n$.
- Therefore we conclude $\forall x P(x)$.

- **Based on the well-ordering property:** Every nonempty set of nonnegative integers has a least element.
Mathematical induction

Example: Prove the sum of first $n$ odd integers is $n^2$.
  i.e. $1 + 3 + 5 + 7 + ... + (2n - 1) = n^2$ for all positive integers.

Proof:
• What is $P(n)$? $P(n): 1 + 3 + 5 + 7 + ... + (2n - 1) = n^2$

Basic Step
Mathematical induction

**Example:** Prove the sum of first n odd integers is $n^2$.

i.e. $1 + 3 + 5 + 7 + \ldots + (2n - 1) = n^2$ for all positive integers.

**Proof:**

- What is $P(n)$?
  - $P(n)$: $1 + 3 + 5 + 7 + \ldots + (2n - 1) = n^2$

**Basis Step**  Show $P(1)$ is true

- Trivial: $1 = 1^2$

**Inductive Step**  Show if $P(n)$ is true then $P(n+1)$ is true for all $n$.

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**Mathematical induction**

**Example:** Prove the sum of first \( n \) odd integers is \( n^2 \).

i.e. \( 1 + 3 + 5 + 7 + \ldots + (2n - 1) = n^2 \) for all positive integers.

**Proof:**

- What is \( P(n) \)? \( P(n) \): \( 1 + 3 + 5 + 7 + \ldots + (2n - 1) = n^2 \)

**Basis Step** Show \( P(1) \) is true

- Trivial: \( 1 = 1^2 \)

**Inductive Step** Show if \( P(n) \) is true then \( P(n+1) \) is true for all \( n \).

- Suppose \( P(n) \) be true, that is \( 1 + 3 + 5 + 7 + \ldots + (2n - 1) = n^2 \)

- Show \( P(n+1) \): \( 1 + 3 + 5 + 7 + \ldots + (2n - 1) + (2n + 1) = (n+1)^2 \) follows:

- \( \underbrace{1 + 3 + 5 + 7 + \ldots + (2n - 1)}_{n^2} + (2n + 1) = (n+1)^2 \)

**Correctness of the mathematical induction**

Suppose \( P(1) \) is true and \( P(n) \) \( \rightarrow P(n+1) \) is true for all positive integers \( n \). Want to show \( \forall x \ P(x) \).
Correctness of the mathematical induction

Suppose $P(1)$ is true and $P(n) \rightarrow P(n+1)$ is true for all positive integers $n$. Want to show $\forall x \ P(x)$.

Assume there is at least one $n$ such that $P(n)$ is false. Let $S$ be the set of nonnegative integers where $P(n)$ is false. Thus $S \neq \emptyset$.

Well-Ordering Property: Every nonempty set of nonnegative integers has a least element.
Correctness of the mathematical induction

Suppose $P(1)$ is true and $P(n) \rightarrow P(n+1)$ is true for all positive integers $n$. Want to show $\forall x \ P(x)$.

Assume there is at least one $n$ such that $P(n)$ is false. Let $S$ be the set of nonnegative integers where $P(n)$ is false. Thus $S \neq \emptyset$.

**Well-Ordering Property:** Every nonempty set of nonnegative integers has a least element.

**By the Well-Ordering Property,** $S$ has a least member, say $k$. $k > 1$, since $P(1)$ is true. This implies $k - 1 > 0$ and $P(k-1)$ is true (since remember $k$ is the smallest integer where $P(k)$ is false).

Now:

$P(k-1) \rightarrow P(k)$ is true

thus, $P(k)$ must be true (a contradiction).

- Therefore $\forall x \ P(x)$. 

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Correctness of the mathematical induction

Suppose $P(1)$ is true and $P(n) \rightarrow P(n+1)$ is true for all positive integers $n$. Want to show $\forall x \ P(x)$.

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- Therefore $\forall x \ P(x)$. 

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Mathematical induction

Example: Prove $n < 2^n$ for all positive integers $n$.

- $P(n): n < 2^n$

**Basis Step:** $1 < 2^1$ (obvious)

**Inductive Step:** If $P(n)$ is true then $P(n+1)$ is true for each $n$.

- Suppose $P(n): n < 2^n$ is true
- Show $P(n+1): n+1 < 2^{n+1}$ is true.
**Mathematical induction**

**Example:** Prove \( n < 2^n \) for all positive integers \( n \).
- \( P(n): \ n < 2^n \)

**Basis Step:** \( 1 < 2^1 \) (obvious)

**Inductive Step:** If \( P(n) \) is true then \( P(n+1) \) is true for each \( n \).
- Suppose \( P(n): \ n < 2^n \) is true
- Show \( P(n+1): \ n+1 < 2^{n+1} \) is true.
  \[
  n + 1 < 2^n + 1
  < 2^n + 2^n
  = 2^n (1 + 1)
  = 2^n (2)
  = 2^{n+1}
  \]

**Mathematical induction**

**Example:** Prove \( n^3 - n \) is divisible by 3 for all positive integers.
- \( P(n): \ n^3 - n \) is divisible by 3

**Basis Step:** \( P(1): \ 1^3 - 1 = 0 \) is divisible by 3 (obvious)
Mathematical induction

Example: Prove $n^3 - n$ is divisible by 3 for all positive integers.

• $P(n)$: $n^3 - n$ is divisible by 3

Basis Step: $P(1)$: $1^3 - 1 = 0$ is divisible by 3 (obvious)

Inductive Step: If $P(n)$ is true then $P(n+1)$ is true for each positive integer.

• Suppose $P(n)$: $n^3 - n$ is divisible by 3 is true.
• Show $P(n+1)$: $(n+1)^3 - (n+1)$ is divisible by 3.

$(n+1)^3 - (n+1) = n^3 + 3n^2 + 3n + 1 - n - 1$

$\begin{align*}
&= n^3 - n + 3n^2 + 3n \\
&= (n^3 - n) + 3(n^2 + n)
\end{align*}$

$\therefore$ divisible by 3      divisible by 3
Strong induction

- The regular induction:
  - uses the basic step \( P(1) \) and
  - inductive step \( P(n-1) \rightarrow P(n) \)

- Strong induction uses:
  - Uses the basis step \( P(1) \) and
  - inductive step \( P(1) \) and \( P(2) \) \( \ldots \) \( P(n-1) \rightarrow P(n) \)

Example: Show that a positive integer greater than 1 can be written as a product of primes.

Strong induction

Example: Show that a positive integer greater than 1 can be written as a product of primes.

Assume \( P(n) \): an integer \( n \) can be written as a product of primes.

Basis step: \( P(2) \) is true

Inductive step: Assume true for \( P(2), P(3), \ldots P(n) \)

Show that \( P(n+1) \) is true as well.
**Strong induction**

**Example:** Show that a positive integer greater than 1 can be written as a product of primes.

Assume P(n): an integer n can be written as a product of primes.

**Basis step:** P(2) is true

**Inductive step:** Assume true for P(2), P(3), … P(n)

Show that P(n+1) is true as well.

2 **Cases:**

- If n+1 is a prime then P(n+1) is trivially true
- If n+1 is a composite then it can be written as a product of two integers (n+1) = a*b such that 1 < a, b < n+1
**Strong induction**

**Example:** Show that a positive integer greater than 1 can be written as a product of primes.

Assume P(n): an integer n can be written as a product of primes.

**Basis step:** P(2) is true

**Inductive step:** Assume true for P(2), P(3), … P(n)

Show that P(n+1) is true as well.

2 Cases:

- If n+1 is a prime then P(n+1) is trivially true
- If n+1 is a composite then it can be written as a product of two integers (n+1) = a*b such that 1 < a, b < n+1
- From the assumption P(a) and P(b) holds.
- Thus, n+1 can be written as a product of primes
- **End of proof**