First-order logic

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Administration

- PS-5 is out

Midterm:
- October 24, 2005
- In class
- Closed book
- Covers:
  - Search, Knowledge Representation and Planning
Limitations of propositional logic

World we want to represent and reason about consists of a number of objects with variety of properties and relations among them.

Propositional logic:
- Represents statements about the world without reflecting this structure and without modeling these entities explicitly.

Consequence:
- Some knowledge is hard or impossible to encode in the propositional logic.
- Two cases that are hard to represent:
  - Statements about similar objects, relations
  - Statements referring to groups of objects.

First-order logic (FOL)

- More expressive than propositional logic.

- Eliminates deficiencies of PL by:
  - Representing objects, their properties, relations and statements about them;
  - Introducing variables that refer to an arbitrary objects and can be substituted by a specific object;
  - Introducing quantifiers allowing us to make statements over groups objects without the need to represent each of them separately.
Logic

Logic is defined by:

- **A set of sentences**
  - A sentence is constructed from a set of primitives according to syntax rules.
- **A set of interpretations**
  - An interpretation gives a semantic to primitives. It associates primitives with objects, values in the real world.
- **The valuation (meaning) function** $V$
  - Assigns a truth value to a given sentence under some interpretation.
  $$V : \text{sentence } \times \text{interpretation} \rightarrow \{\text{True }, \text{False }\}$$

First-order logic. Syntax.

**Term** - syntactic entity for representing objects

**Terms in FOL:**
- **Constant symbols**: represent specific objects
  - Examples: John, France, car89

- **Variables** – represents object of specific type (defined by the universe of discourse)
  - Examples: x, y
  - (universe of discourse can be people, students, cars)

- **Functions** applied to one or more terms
  - Examples: $\text{father-of}(\text{John})$, $\text{father-of(father-of(John))}$
**First order logic. Syntax.**

- Terms do not define propositions (they cannot be evaluated to True or False)

**Sentences in FOL define propositions:**

- **Atomic sentences:**
  - A **predicate symbol** applied to 0 or more terms
    
    **Examples:**
    
    - `Red(car12),`
    - `Sister(Amy, Jane);`
    - `Manager(father-of(John));`
  
  - `t1 = t2` **equivalence** of terms

  **Example:**
  
  - `John = father-of(Peter)`

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**First order logic. Syntax.**

**Sentences in FOL:**

- **Complex sentences:**
  - Assume \( \phi, \psi \) are sentences in FOL. Then:
    - \( (\phi \land \psi) \) \( (\phi \lor \psi) \) \( (\phi \Rightarrow \psi) \) \( (\phi \iff \psi) \) \( \neg \psi \)
    
    and
  
  - \( \forall x \phi \) \( \exists y \phi \)
    
    are sentences

**Symbols** `\exists, \forall`

- stand for the **existential** and the **universal** quantifier
Semantics. Interpretation.

An interpretation $I$ is defined by a mapping to the \textbf{domain of discourse} $D$ or relations on $D$

- \textbf{domain of discourse}: a set of objects in the world we represent and refer to;

\textbf{An interpretation $I$ maps}:

- Constant symbols to objects in $D$
  \[ I(\text{John}) = \] \[ I(\text{Paul}) = \]

- Predicate symbols to relations, properties on $D$
  \[ I(\text{brother}) = \{ \langle \text{John} \text{ Paul} \rangle; \langle \text{Paul} \text{ John} \rangle; \ldots \} \]

- Function symbols to functional relations on $D$
  \[ I(\text{father-of}) = \{ \langle \text{John} \rightarrow \text{Paul} \rangle; \langle \text{Paul} \rightarrow \text{John} \rangle; \ldots \} \]

Semantics of sentences.

\textbf{Meaning (evaluation) function}:

\[ V : \text{sentence} \times \text{interpretation} \rightarrow \{ \text{True} , \text{False} \} \]

A \textbf{predicate} $\text{predicate}(\text{term-1, term-2, term-3, term-n})$ is true for the interpretation $I$, iff the objects referred to by $\text{term-1, term-2, term-3, term-n}$ are in the relation referred to by $\text{predicate}$

\[ I(\text{John}) = \] \[ I(\text{Paul}) = \]

\[ I(\text{brother}) = \{ \langle \text{John} \text{ Paul} \rangle; \langle \text{Paul} \text{ John} \rangle; \ldots \} \]

\[ \text{brother(John, Paul)} = \langle \text{John} \text{ Paul} \rangle \quad \text{in} \ I(\text{brother}) \]

\[ V(\text{brother(John, Paul)}, I) = \text{True} \]
Semantics of sentences.

- **Equality**
  \[ V(\text{term-1} = \text{term-2}, I) = \text{True} \]
  If \( I(\text{term-1}) = I(\text{term-2}) \)

- **Boolean expressions**: standard
  
  E.g. \( V(\text{sentence-1} \lor \text{sentence-2}, I) = \text{True} \)
  
  If \( V(\text{sentence-1},I) = \text{True} \) or \( V(\text{sentence-2},I) = \text{True} \)

- **Quantifications**
  
  \[ V(\forall x \phi, I) = \text{True} \]
  
  If \( \text{for all } d \in D \ V(\phi, I[x/d]) = \text{True} \)

  \[ V(\exists x \phi, I) = \text{True} \]
  
  If \( \text{there is a } d \in D \), s.t. \( V(\phi, I[x/d]) = \text{True} \)

Sentences with quantifiers

- **Universal quantification**
  
  *All Upitt students are smart*

  - Assume the universe of discourse of x are Upitt students
Sentences with quantifiers

• **Universal quantification**

   *All Upitt students are smart*

• Assume the universe of discourse of x are Upitt students

   \[ \forall x \text{ smart}(x) \]
Sentences with quantifiers

• **Universal quantification**

  \[ \forall x \text{ smart}(x) \]
  
  *All Upitt students are smart*

• Assume the universe of discourse of x are Upitt students

  \[ \forall x \text{ smart}(x) \]

• Assume the universe of discourse of x are students

  \[ \forall x \text{ at}(x, \text{Upitt} ) \Rightarrow \text{smart}(x) \]

• Assume the universe of discourse of x are people
Sentences with quantifiers

- **Universal quantification**

  *All Upitt students are smart*

  

  - Assume the universe of discourse of x are Upitt students
    
    \[ \forall x \text{ smart}(x) \]

  - Assume the universe of discourse of x are students
    
    \[ \forall x \text{ at}(x, \text{Upitt}) \implies \text{smart}(x) \]

  - Assume the universe of discourse of x are people
    
    \[ \forall x \text{ student}(x) \land \text{at}(x, \text{Upitt}) \implies \text{smart}(x) \]

  *Typically the universal quantifier connects with an implication*
Sentences with quantifiers

• Existential quantification

*Someone at CMU is smart*

• Assume the universe of discourse of x are CMU affiliates

\[ \exists x \ (at(x, CMU) \land \text{smart}(x)) \]
Sentences with quantifiers

• **Existential quantification**

Someone at CMU is smart

• Assume the universe of discourse of x are CMU affiliates

\[ \exists x \ at(x,CMU) \land smart(x) \]

• Assume the universe of discourse of x are people

\[ \exists x \ at(x,CMU) \land smart(x) \]
Sentences with quantifiers

- **Existential quantification**

  Someone at CMU is smart

  \[ \exists x \at(x, CMU) \land \text{smart}(x) \]

- Assume the universe of discourse of \( x \) are CMU affiliates

  \[ \exists x \at(x, CMU) \land \text{smart}(x) \]

- Assume the universe of discourse of \( x \) are people

  \[ \exists x \at(x, CMU) \land \text{smart}(x) \]

  *Typically the existential quantifier connects with a conjunction*

Order of quantifiers

- **Order of quantifiers of the same type does not matter**

  For all \( x \) and \( y \), if \( x \) is a parent of \( y \) then \( y \) is a child of \( x \)

  \[ \forall x, y \parent (x, y) \Rightarrow \child (y, x) \]

  \[ \forall y, x \parent (x, y) \Rightarrow \child (y, x) \]

- **Order of different quantifiers changes the meaning**

  \[ \forall x \exists y \loves (x, y) \]
Order of quantifiers

- **Order of quantifiers of the same type does not matter**
  
  For all $x$ and $y$, if $x$ is a parent of $y$ then $y$ is a child of $x$
  
  \[
  \forall x, y \text{ parent } (x, y) \implies \text{ child } (y, x) \\
  \forall y, x \text{ parent } (x, y) \implies \text{ child } (y, x)
  \]

- **Order of different quantifiers changes the meaning**
  
  \[
  \forall x \exists y \text{ loves } (x, y) \\
  \text{Everybody loves somebody}
  \]
  
  \[
  \exists y \forall x \text{ loves } (x, y)
  \]

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Order of quantifiers

- **Order of quantifiers of the same type does not matter**

  For all $x$ and $y$, if $x$ is a parent of $y$ then $y$ is a child of $x$

  \[
  \forall x, y \text{ parent } (x, y) \implies \text{ child } (y, x) \\
  \forall y, x \text{ parent } (x, y) \implies \text{ child } (y, x)
  \]

- **Order of different quantifiers changes the meaning**

  \[
  \forall x \exists y \text{ loves } (x, y) \\
  \text{Everybody loves somebody}
  \]

  \[
  \exists y \forall x \text{ loves } (x, y) \\
  \text{There is someone who is loved by everyone}
  \]
Connections between quantifiers

Everyone likes ice cream

\( \forall x \text{ likes } (x, \text{IceCream}) \)
Connections between quantifiers

\[ \forall x \text{ likes } (x, \text{IceCream}) \]

Is it possible to convey the same meaning using an existential quantifier?

\[ \neg \exists x \neg \text{likes } (x, \text{IceCream}) \]

A universal quantifier in the sentence can be expressed using an existential quantifier !!!
Connections between quantifiers

Someone likes ice cream

?  

Is it possible to convey the same meaning using a universal quantifier?

∃x likes (x, IceCream)
Connections between quantifiers

Someone likes ice cream

\[ \exists x \text{ likes } (x, \text{IceCream}) \]

Is it possible to convey the same meaning using a universal quantifier?

Not everyone does not like ice cream

\[ \neg \forall x \neg \text{likes } (x, \text{IceCream}) \]

An existential quantifier in the sentence can be expressed using a universal quantifier !!!

Representing knowledge in FOL

Example:

Kinship domain

- **Objects:** people
  - John, Mary, Jane, …
- **Properties:**
  - Male \((x)\), Female \((x)\)
- **Relations:**
  - parenthood, brotherhood, marriage
  - Parent \((x, y)\), Brother \((x, y)\), Spouse \((x, y)\)
- **Functions:** mother-of (one for each person \(x\))
  - MotherOf \((x)\)
Kinship domain in FOL

**Relations between predicates and functions:** write down what we know about them; how relate to each other.

- Male and female are disjoint categories
  \[ \forall x \text{ Male} (x) \iff \neg \text{Female} (x) \]
- Parent and child relations are inverse
  \[ \forall x, y \text{ Parent} (x, y) \iff \text{Child} (y, x) \]
- A grandparent is a parent of parent
  \[ \forall g, c \text{ Grandparent}(g, c) \iff \exists p \text{ Parent}(g, p) \land \text{Parent}(p, c) \]
- A sibling is another child of one’s parents
  \[ \forall x, y \text{ Sibling} (x, y) \iff (x \neq y) \land \exists p \text{ Parent} (p, x) \land \text{ Parent} (p, y) \]
- And so on ….

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Inference in First order logic
Logical inference in FOL

Logical inference problem:
• Given a knowledge base KB (a set of sentences) and a sentence $\alpha$, does the KB semantically entail $\alpha$?

$$KB \models \alpha$$

In other words: In all interpretations in which sentences in the KB are true, is also $\alpha$ true?

Logical inference problem in the first-order logic is undecidable !!!: No procedure that can decide the entailment for all possible input sentences in a finite number of steps.

Logical inference problem in the Propositional logic

Computational procedures that answer:

$$KB \models \alpha$$

Three approaches:
• Truth-table approach
• Inference rules
• Conversion to the inverse SAT problem
  – Resolution-refutation
Inference in FOL: Truth table

• Is the Truth-table approach a viable approach for the FOL?

? 

Inference in FOL: Truth table approach

• Is the Truth-table approach a viable approach for the FOL?

• NO! 

• Why?
• It would require us to enumerate and list all possible interpretations I
• I = (assignments of symbols to objects, predicates to relations and functions to relational mappings)
• Simply there are too many interpretations
Inference in FOL: Inference rules

• Is the Inference rule approach a viable approach for the FOL?

• Yes.

• The inference rules represent sound inference patterns one can apply to sentences in the KB

• What is derived follows from the KB

• Caveat: we need to add rules for handling quantifiers
Inference rules

• Inference rules from the propositional logic:
  – Modus ponens
    \[
    \frac{A \Rightarrow B, \ A}{B}
    \]
  – Resolution
    \[
    \frac{A \lor B, \ \neg B \lor C}{A \lor C}
    \]
  – and others: And-introduction, And-elimination, Or-introduction, Negation elimination

• Additional inference rules are needed for sentences with quantifiers and variables
  – Must involve variable substitutions

Sentences with variables

First-order logic sentences can include variables.

• Variable is:
  – Bound – if it is in the scope of some quantifier
    \[
    \forall x \ P(x)
    \]
  – Free – if it is not bound.
    \[
    \exists x \ P(y) \land Q(x) \quad y \text{ is free}
    \]

• Sentence (formula) is:
  – Closed – if it has no free variables
    \[
    \forall y \exists x \ P(y) \Rightarrow Q(x)
    \]
  – Open – if it is not closed
  – Ground – if it does not have any variables
    \[
    Likes(John, Jane)
    \]
Variable substitutions

- Variables in the sentences can be substituted with terms.
  (terms = constants, variables, functions)

- **Substitution:**
  - Is represented by a mapping from variables to terms
    \[ \{x_1 / t_1, x_2 / t_2, \ldots\} \]
  - Application of the substitution to sentences
    \[
    \text{SUBST}(\{x / Sam, y / Pam\}, \text{Likes}(x,y)) = \text{Likes}(Sam, Pam)
    \]
    \[
    \text{SUBST}(\{x / z, y / fatherof(John)\}, \text{Likes}(x,y)) = \text{Likes}(z, fatherof(John))
    \]

Inference rules for quantifiers

- **Universal elimination**
  \[
  \frac{\forall x \, \phi(x)}{\phi(a)} \quad a \text{ - is a constant symbol}
  \]
  - Substitutes a variable with a constant symbol
    \[
    \forall x \, \text{Likes}(x, \text{IceCream}) \quad \text{Likes}(Ben, \text{IceCream})
    \]

- **Existential elimination.**
  \[
  \frac{\exists x \, \phi(x)}{\phi(a)}
  \]
  - Substitutes a variable with a constant symbol that does not appear elsewhere in the KB
    \[
    \exists x \, \text{Kill}(x, \text{Victim}) \quad \text{Kill}(\text{Murderer, Victim})
    \]