Knowledge Representation.
Propositional logic

Milos Hauskrecht
milos@cs.pitt.edu
5329 Sennott Square

Knowledge-based agent

<table>
<thead>
<tr>
<th>Knowledge base</th>
<th>Inference engine</th>
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</table>

- **Knowledge base (KB):**
  - Knowledge that describe facts about the world in some formal (representational) language
  - **Domain specific**
- **Inference engine:**
  - A set of procedures that use the representational language to infer new facts from known ones or answer a variety of KB queries. **Inferences typically require search.**
  - **Domain independent**
Example: MYCIN

- MYCIN: an expert system for diagnosis of bacterial infections
- **Knowledge base** represents
  - **Facts** about a specific patient case
  - **Rules** describing relations between entities in the bacterial infection domain

| If | 1. The stain of the organism is gram-positive, and  
|    | 2. The morphology of the organism is coccus, and  
|    | 3. The growth conformation of the organism is chains |
|    | Then the identity of the organism is streptococcus |

- **Inference engine:**
  - manipulates the facts and known relations to answer diagnostic queries (consistent with findings and rules)

Knowledge representation

- **Objective:** express the knowledge about the world in a computer-tractable form
- **Knowledge representation languages (KRLs)**

  Key aspects:
  - **Syntax:** describes how sentences in KRL are formed in the language
  - **Semantics:** describes the meaning of sentences, what is it the sentence refers to in the real world
  - **Computational aspect:** describes how sentences and objects in KRL are manipulated in concordance with semantic conventions

Many KB systems rely on and implement some variant of logic
Logic

A formal language for expressing knowledge and for making logical inferences

**Defined by:**

- **A set of sentences:** A *sentence* is constructed from a set of *primitives* according to *syntactic rules*
- **A set of interpretations:** An interpretation I gives a *semantic to primitives*. It associates primitives with objects or values
  - I: primitives $\rightarrow$ objects/values
- **The valuation (meaning) function $V$:**
  - Assigns a value (typically the truth value) to a given *sentence* under some interpretation
    $V : \text{sentence} \times \text{interpretation} \rightarrow \{True, False\}$

Propositional logic

- **The simplest logic**

- **Definition:**
  - A *proposition* is a statement that is either true or false.

- **Examples:**
  - Pitt is located in the Oakland section of Pittsburgh.
    - (T)
Propositional logic

• The simplest logic

• **Definition:**
  – A proposition is a statement that is either true or false.

• Examples:
  – Pitt is located in the Oakland section of Pittsburgh.
    • (T)
  – $5 + 2 = 8$.
    • ?
Propositional logic

• The simplest logic

• **Definition:**
  – A *proposition* is a statement that is either true or false.

• Examples:
  – Pitt is located in the Oakland section of Pittsburgh.
    • (T)
  – $5 + 2 = 8$.
    • (F)
  – It is raining today.
    • (either T or F)

Propositional logic

• Examples (cont.):
  – How are you?
    • ?
Propositional logic

• Examples (cont.):
  – How are you?
    • a question is not a proposition
  – $x + 5 = 3$
    • ?

• Examples (cont.):
  – How are you?
    • a question is not a proposition
  – $x + 5 = 3$
    • since $x$ is not specified, neither true nor false
  – $2$ is a prime number.
    • ?
Propositional logic

• Examples (cont.):
  – How are you?
    • a question is not a proposition
  – $x + 5 = 3$
    • since $x$ is not specified, neither true nor false
  – 2 is a prime number.
    • (T)

Propositional logic. Syntax

• Formally propositional logic $P$:
  – Is defined by Syntax+interpretation+semantics of $P$

Syntax:
• Symbols (alphabet) in $P$:
  – Constants: True, False
  – Propositional symbols
    Examples:
    • $P$
      • Pitt is located in the Oakland section of Pittsburgh.,
      • It rains outside, etc.
  – A set of connectives:
    $\neg, \land, \lor, \Rightarrow, \Leftrightarrow$
Propositional logic. Syntax

Sentences in the propositional logic:

- **Atomic sentences:**
  - Constructed from constants and propositional symbols
  - True, False are (atomic) sentences
  - $P, Q$ or *Light in the room is on, It rains outside* are (atomic) sentences

- **Composite sentences:**
  - Constructed from valid sentences via logical connectives
  - If $A, B$ are sentences then
    - $\neg A$ ( $A \land B$ ) $A \lor B$ $A \Rightarrow B$ $A \Leftrightarrow B$
    - or $A \lor B \land$ ($A \lor \neg B$)
    - are sentences

Propositional logic. Semantics.

The semantic gives the meaning to sentences.

The semantics in the propositional logic is defined by:

1. **Interpretation of propositional symbols and constants**
   - Semantics of atomic sentences

2. **Through the meaning of logical connectives**
   - Meaning (semantics) of composite sentences
Semantic: propositional symbols

A propositional symbol
• a statement about the world that is either true or false
Examples:
– Pitt is located in the Oakland section of Pittsburgh
– It rains outside
– Light in the room is on
• An interpretation maps symbols to one of the two values: True (T), or False (F), depending on whether the symbol is satisfied in the world

I: Light in the room is on -> True, It rains outside -> False
I’: Light in the room is on -> False, It rains outside -> False

Semantic: propositional symbols

The meaning (value) of the propositional symbol for a specific interpretation is given by its interpretation

I: Light in the room is on -> True, It rains outside -> False
V(Light in the room is on, I) = True
V( It rains outside, I) = False

I’: Light in the room is on -> False, It rains outside -> False
V(Light in the room is on, I’) = False

<table>
<thead>
<tr>
<th>Interpretations</th>
<th>Meanings (values)</th>
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Semantics: constants

- The meaning (truth) of constants:
  - True and False constants are always (under any interpretation) assigned the corresponding True, False value

\[
V(\text{True}, I) = \text{True} \\
V(\text{False}, I) = \text{False}
\]

For any interpretation \( I \)

Semantics: composite sentences.

- The meaning (truth value) of complex propositional sentences.
  - Determined using the standard rules of logic:

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<th>( P )</th>
<th>( Q )</th>
<th>( \neg P )</th>
<th>( P \land Q )</th>
<th>( P \lor Q )</th>
<th>( P \Rightarrow Q )</th>
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Translation

Translation of English sentences to propositional logic:
(1) identify atomic sentences that are propositions
(2) Use logical connectives to translate more complex composite sentences that consist of many atomic sentences

Assume the following sentence:
• It is not sunny this afternoon and it is colder than yesterday.

Atomic sentences:
• p = It is sunny this afternoon
• q = it is colder than yesterday

Translation: \( \neg p \land q \)

Translation

Assume the following sentences:
• It is not sunny this afternoon and it is colder than yesterday.
• We will go swimming only if it is sunny.
• If we do not go swimming then we will take a canoe trip.
• If we take a canoe trip, then we will be home by sunset.

Denote:
• p = It is sunny this afternoon
• q = it is colder than yesterday
• r = We will go swimming
• s= we will take a canoe trip
• t= We will be home by sunset
Translation

Assume the following sentences:
• It is not sunny this afternoon and it is colder than yesterday. \( \neg p \land q \)
• We will go swimming only if it is sunny.
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Denote:
• \( p \) = It is sunny this afternoon
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• \( t \) = We will be home by sunset
Translation

Assume the following sentences:
• It is not sunny this afternoon and it is colder than yesterday. \( \neg p \land q \)
• We will go swimming only if it is sunny. \( r \rightarrow p \)
• If we do not go swimming then we will take a canoe trip. \( \neg r \rightarrow s \)
• If we take a canoe trip, then we will be home by sunset. \( s \rightarrow t \)

Denote:
• \( p \) = It is sunny this afternoon
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• \( r \) = We will go swimming
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• \( t \) = We will be home by sunset
Contradiction and Tautology

Some composite sentences may always (under any interpretation) evaluate to a single truth value:

- **Contradiction** (always *False*)
  \[ P \land \neg P \]

- **Tautology** (always *True*)
  \[ P \lor \neg P \]

\[ \neg(P \lor Q) \iff (\neg P \land \neg Q) \]
\[ \neg(P \land Q) \iff (\neg P \lor \neg Q) \]
\] DeMorgan’s Laws

Model, validity and satisfiability

- An interpretation is a model for a set of sentences if it assigns true to each sentence in the set.
- **Example 1:**
  - Primitives: \( P, Q \)  
  - Sentence: \( P \lor Q \)
  - Interpretations:
    - Model?
      - \( P \to True, Q \to True \)

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- Primitives: P,Q
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• Example 1:
  – Primitives: P, Q  
  – Sentence: \( P \lor Q \)  
  – Interpretations:  
    • \( P \rightarrow True, Q \rightarrow True \)  
      Model: Yes  
    • \( P \rightarrow True, Q \rightarrow False \)  
      Model: Yes  
    • \( P \rightarrow False, Q \rightarrow False \)  
      Model: No

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Model, validity and satisfiability

• An interpretation is a model for a set of sentences if it assigns true to each sentence in the set.

• Example 2:
  – Primitives: P, Q  
  – Sentences:  
    \( P \lor Q \)  
    \( (P \lor Q) \land \neg Q \)  
    \( ((P \lor Q) \land \neg Q) \Rightarrow P \)  
  – Is there a model?

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Model, validity and satisfiability

• An interpretation is a model for a set of sentences if it assigns true to each sentence in the set.

• Example 2:
  – Primitives: P, Q
  – Sentences: $P \lor Q$
    
    $(P \lor Q) \land \neg Q$
    
    $((P \lor Q) \land \neg Q) \Rightarrow P$

  – Is there a model? Yes $P \rightarrow True$ $Q \rightarrow False$

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Model, validity and satisfiability

• An interpretation is a model for a set of sentences if it assigns true to each sentence in the set.

• A sentence is satisfiable if it has a model;
  – There is at least one interpretation under which the sentence can evaluate to True

• Example:
  – Sentence: $(P \lor Q) \land \neg Q$
  – Satisfiable?

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- A sentence is **satisfiable** if it has a model;
  - There is at least one interpretation under which the sentence can evaluate to True.
- **Example:**
  - Sentence: \((P \lor Q) \land \neg Q\)
  - Satisfiable? Yes  True for \(P \rightarrow True, Q \rightarrow False\)

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- A sentence is **satisfiable** if it has a model;
  - There is at least one interpretation under which the sentence can evaluate to True.
- A sentence is **valid** if it is **True in all interpretations**
  - i.e., if its negation is **not satisfiable** (leads to contradiction)

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<th></th>
<th>P</th>
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valid sentence?

Entailment

- Entailment reflects the relation of one fact in the world following from the others

- Entailment $KB \models \alpha$
- Knowledge base KB entails sentence $\alpha$ if and only if $\alpha$ is true in all worlds (interpretations) where KB is true
Sound and complete inference

**Inference** is a process by which new sentences are derived from existing sentences in the KB

- the inference process is implemented on a computer

Assume an **inference procedure** \(i\) that

- derives a sentence \(\alpha\) from the KB: \(KB \vdash_i \alpha\)

**Properties of the inference procedure in terms of entailment**

- **Soundness:** An inference procedure is **sound**
  
  \[
  \text{If } KB \vdash_i \alpha \text{ then it is true that } KB \models \alpha
  \]

- **Completeness:** An inference procedure is **complete**
  
  \[
  \text{If } KB \models \alpha \text{ then it is true that } KB \vdash_i \alpha
  \]

Logical inference problem

**Logical inference problem:**

**Given:**

- a knowledge base KB (a set of sentences) and
- a sentence \(\alpha\) (called a theorem),

- **Does a KB semantically entail** \(\alpha\)? \(KB \models \alpha\)?

In other words: In all interpretations in which sentences in the KB are true, is also \(\alpha\) true?

**Question:** Is there a procedure (program) that can decide this problem in a finite number of steps?

**Answer:** Yes. Logical inference problem for the propositional logic is **decidable.**