CS 2750 Foundations of AI
Lecture 17

Bayesian belief networks

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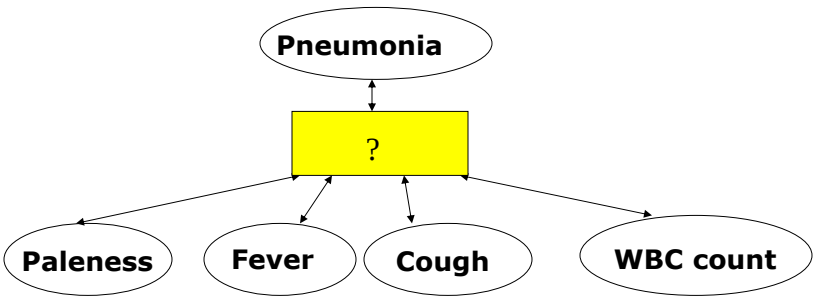
Modeling the uncertainty.

Key challenges:

- How to represent the relations in the presence of uncertainty?
- How to manipulate such knowledge to make inferences?
  - **Humans can reason with uncertainty.**



Modeling uncertainty with probabilities

Probabilistic extension of propositional logic

- **Propositions:**
  - statements about the world
  - Represented by the assignment of values to **random variables**
- **Random variables:**

! – **Boolean** $Pneumonia$ is either $True, False$

Random variable **Values**

! – **Multi-valued** $Pain$ is one of $\{Nopain, Mild, Moderate, Severe\}$

Random variable **Values**

– **Continuous** $HeartRate$ is a value in $<0;180>$

Random variable **Values**

Probability distribution

Defines probability for **all possible value assignments**

Example 1:

$P(Pneumonia = True) = 0.001$

$P(Pneumonia = False) = 0.999$

$Pneumonia$	$\mathbf{P}(Pneumonia)$
$True$	0.001
$False$	0.999

$$P(Pneumonia = True) + P(Pneumonia = False) = 1$$

Probabilities sum to 1 !!!

Example 2:

$P(WBCcount = high) = 0.005$

$P(WBCcount = normal) = 0.993$

$P(WBCcount = high) = 0.002$

$WBCcount$	$\mathbf{P}(WBCcount)$
$high$	0.005
$normal$	0.993
low	0.002

Joint probability distribution

Joint probability distribution (for a set variables)

- Defines probabilities for **all possible assignments of values to variables in the set**

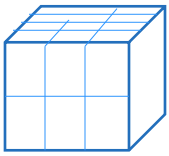
Example: Assume variables:

Pneumonia (2 values)

WBCcount (3 values)

Pain (4 values)

$\mathbf{P}(pneumonia, WBCcount, Pain)$ is represented by $2 \times 3 \times 4$ array



Joint probabilities

Marginalization

- reduces the dimension of the joint distribution
- Sums variables out

$\mathbf{P}(pneumonia, WBCcount)$ 2×3 matrix

$\mathbf{P}(Pneumonia)$

		WBCcount			
		high	*normal*	*low*	
Pneumonia	*True*	0.0008	0.0001	0.0001	0.001
	False	0.0042	0.9929	0.0019	0.999
		0.005	0.993	0.002	

$\mathbf{P}(WBCcount)$

Marginalization (here summing of columns or rows)

Marginalization

Marginalization

- reduces the dimension of the joint distribution

$$P(X_1, X_2, \ldots X_{n-1}) = \sum_{\{X_n\}} P(X_1, X_2, \ldots X_{n-1}, X_n)$$

- We can continue doing this

$$P(X_2, \ldots X_{n-1}) = \sum_{\{X_1, X_n\}} P(X_1, X_2, \ldots X_{n-1}, X_n)$$

What is the maximal joint probability distribution?

- Full joint probability

Full joint distribution

The joint distribution for all variables in the problem

– It defines the complete probability model for the problem

Example: pneumonia diagnosis

Variables: *Pneumonia, Fever, Paleness, WBCcount, Cough*

Full joint defines the probability for all possible assignments of values to these variables

$$P(Pneumonia = T, WBCcount = High, Fever = T, Cough = T, Paleness = T)$$

$$P(Pneumonia = T, WBCcount = High, Fever = T, Cough = T, Paleness = F)$$

$$P(Pneumonia = T, WBCcount = High, Fever = T, Cough = F, Paleness = T)$$

$\ldots$ etc

- **How many probabilities are there?**

Full joint distribution

The joint distribution for all variables in the problem

– It defines the complete probability model for the problem

Example: pneumonia diagnosis

Variables: *Pneumonia, Fever, Paleness, WBCcount, Cough*

Full joint defines the probability for all possible assignments of values to these variables

$$P(Pneumonia = T, WBCcount = High, Fever = T, Cough = T, Paleness = T)$$

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$$P(Pneumonia = T, WBCcount = High, Fever = T, Cough = F, Paleness = T)$$

$$\ldots \quad \text{etc}$$

- **How many probabilities are there?**

Joint probabilities and independence

- **Is it possible to recover the full joint from the joint probabilities over a subset of variables?**
- Only if the variables are independent !!!

$\mathbf{P}(pneumonia, WBCcount)$ $\quad 2 \times 3$ matrix

$\mathbf{P}(Pneumonia)$

		WBCcount			
		high	*normal*	*low*	
Pneumonia	*True*	?	?	?	0.001
	False	?	?	?	0.999
		0.005	0.993	0.002	

$\mathbf{P}(WBCcount)$

Conditional probabilities

- **Conditional probability distribution.**

$$P(A \mid B) = \frac{P(A,B)}{P(B)} \text{ s.t. } P(B) \neq 0$$

- **Product rule.** A join probability can be expressed in terms of conditional probabilities

$$P(A,B) = P(A \mid B)P(B)$$

- **Chain rule**. Any joint probability can be expressed as a product of conditionals

$$\begin{aligned} P(X_1, X_2, \ldots X_n) &= P(X_n \mid X_{1,} \ldots X_{n-1}) P(X_{1,} \ldots X_{n-1}) \\ &= P(X_n \mid X_{1,} \ldots X_{n-1}) P(X_{n-1} \mid X_{1,} \ldots X_{n-2}) P(X_{1,} \ldots X_{n-2}) \\ &= \prod_{i=1}^{n} P(X_i \mid X_{1,} \ldots X_{i-1}) \end{aligned}$$

Conditional probabilities

Conditional probability

- Is defined in terms of the joint probability:

$$P(A \mid B) = \frac{P(A,B)}{P(B)} \text{ s.t. } P(B) \neq 0$$

- **Example:**

$$P(pneumonia = true \mid WBCcount = high) = \frac{P(pneumonia = true, WBCcount = high)}{P(WBCcount = high)}$$

$$P(pneumonia = false \mid WBCcount = high) = \frac{P(pneumonia = false, WBCcount = high)}{P(WBCcount = high)}$$

Conditional probabilities

Conditional probability distribution

- Defines probabilities for all possible assignments, given a fixed assignment to some other variable values

$$P(Pneumonia = true \mid WBCcount = high)$$

$\mathbf{P}(Pneumonia \mid WBCcount)$ 3 element vector of 2 elements

		WBCcount		
		high	*normal*	*low*
Pneumonia	*True*	0.08	0.0001	0.0001
	False	0.92	0.9999	0.9999
		1.0	1.0	1.0

$$P(Pneumonia = true \mid WBCcount = high)$$
$$+ P(Pneumonia = false \mid WBCcount = high)$$

Bayes rule

Conditional probability.

$$P(A \mid B) = \frac{P(A,B)}{P(B)} \qquad P(A,B) = P(B \mid A)P(A)$$

Bayes rule:

$$P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B)}$$

When is it useful?

- When we are interested in computing the diagnostic query from the causal probability

$$P(cause \mid effect) = \frac{P(effect \mid cause)P(cause)}{P(effect)}$$

- **Reason:** It is often easier to assess causal probability
  - E.g. Probability of pneumonia causing fever
    vs. probability of pneumonia given fever

Probabilistic inference

Various inference tasks:

- **Diagnostic task. (from effect to cause)**

$$\mathbf{P}(Pneumonia \mid Fever = T)$$

- **Prediction task. (from cause to effect)**

$$\mathbf{P}(Fever \mid Pneumonia = T)$$

- **Other probabilistic queries** (queries on joint distributions).

$$\mathbf{P}(Fever)$$

$$\mathbf{P}(Fever, ChestPain)$$

Inference

Any query can be computed from the full joint distribution !!!

- **Joint over a subset of variables** is obtained through marginalization

$$P(A=a, C=c) = \sum_i \sum_j P(A=a, B=b_i, C=c, D=d_j)$$

- **Conditional probability over set of variables**, given other variables' values is obtained through marginalization and definition of conditionals

$$P(D=d \mid A=a, C=c) = \frac{P(A=a, C=c, D=d)}{P(A=a, C=c)}$$

$$= \frac{\sum_i P(A=a, B=b_i, C=c, D=d)}{\sum_i \sum_j P(A=a, B=b_i, C=c, D=d_j)}$$

Modeling uncertainty with probabilities

- Defining the **full joint distribution** makes it possible to represent and reason with uncertainty in a uniform way
- We are able to handle an arbitrary probabilistic inference problem

Problems:

- **Space complexity.** To store a full joint distribution we need to remember $O(d^n)$ numbers.

  n – number of random variables, d – number of values
- **Inference (time) complexity.** To compute some queries requires $O(d^n)$ steps.
- **Acquisition problem.** Who is going to define all of the probability entries?

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Modeling uncertainty with probabilities

- **Knowledge based system era (70s – early 80's)**
  - **Extensional non-probabilistic models**
  - Solve the space, time and acquisition bottlenecks in probability-based models
  - Froze the development and advancement of KB systems and contributed to the slow-down of AI in 80s in general
- Breakthrough (late 80s, beginning of 90s)
  - **Bayesian belief networks**
    - Give solutions to the space, acquisition bottlenecks
    - Partial solutions for time complexities
- Bayesian belief network

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Bayesian belief networks (BBNs)

Bayesian belief networks.

- Represent the full joint distribution over the variables more compactly with a **smaller number of parameters**.
- Take advantage of **conditional and marginal independences** among random variables

- **A and B are independent**

$$P(A,B) = P(A)P(B)$$

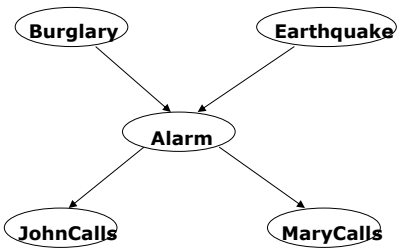
- **A and B are conditionally independent given C**

$$P(A,B \mid C) = P(A \mid C)P(B \mid C)$$
$$P(A \mid C,B) = P(A \mid C)$$

Alarm system example

- Assume your house has an **alarm system** against **burglary**. You live in the seismically active area and the alarm system can get occasionally set off by an **earthquake**. You have two neighbors, **Mary** and **John**, who do not know each other. If they hear the alarm they call you, but this is not guaranteed.
- We want to represent the probability distribution of events:
  - Burglary, Earthquake, Alarm, Mary calls and John calls

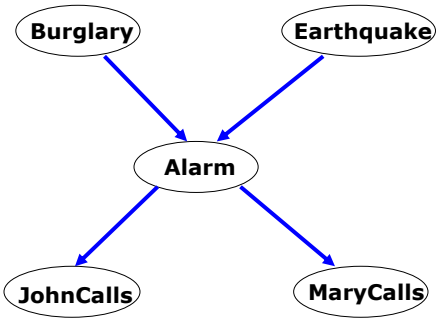
Causal relations



Bayesian belief network

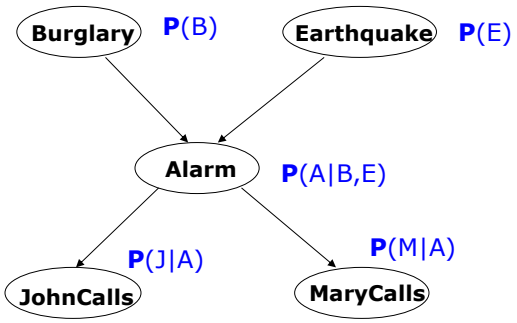
1. **Directed acyclic graph**
   - **Nodes** = random variables
     Burglary, Earthquake, Alarm, Mary calls and John calls
   - **Links** = direct (causal) dependencies between variables.

     The chance of Alarm is influenced by Earthquake, The chance of John calling is affected by the Alarm

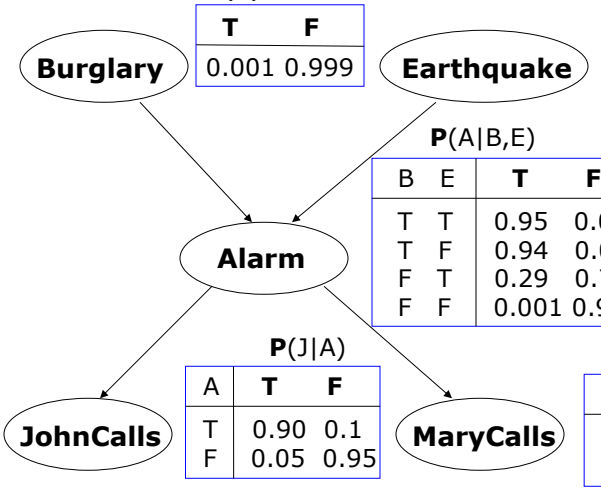


Bayesian belief network

2. **Local conditional distributions**
   - relate variables and their parents



Bayesian belief network



P(B)

T	F
0.001	0.999

P(E)

T	F
0.002	0.998

P(A|B,E)

B	E	T	F
T	T	0.95	0.05
T	F	0.94	0.06
F	T	0.29	0.71
F	F	0.001	0.999

P(J|A)

A	T	F
T	0.90	0.1
F	0.05	0.95

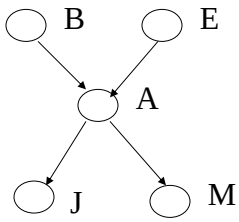
P(M|A)

A	T	F
T	0.7	0.3
F	0.01	0.99

Bayesian belief networks (general)

Two components: $B = (S, \Theta_S)$

- **Directed acyclic graph**
  - Nodes correspond to random variables
  - (Missing) links encode independences



- **Parameters**
  - Local conditional probability distributions for every variable-parent configuration

$$\mathbf{P}(X_i \mid pa(X_i))$$

Where:

$pa(X_i)$ - stand for parents of X_i

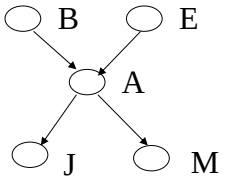
P(A|B,E)

B	E	T	F
T	T	0.95	0.05
T	F	0.94	0.06
F	T	0.29	0.71
F	F	0.001	0.999

Full joint distribution in BBNs

Full joint distribution is defined in terms of local conditional distributions (obtained via the chain rule):

$$\mathbf{P}(X_1, X_2, .., X_n) = \prod_{i=1,..n} \mathbf{P}(X_i \mid pa(X_i))$$



Example:

Assume the following assignment of values to random variables

$B = T, E = T, A = T, J = T, M = F$

Then its probability is:

$$P(B = T, E = T, A = T, J = T, M = F) =$$

$$P(B = T)P(E = T)P(A = T \mid B = T, E = T)P(J = T \mid A = T)P(M = F \mid A = T)$$

Bayesian belief networks (BBNs)

Bayesian belief networks

- Represent the full joint distribution over the variables more compactly using the product of local conditionals.
- **But how did we get to local parameterization?**

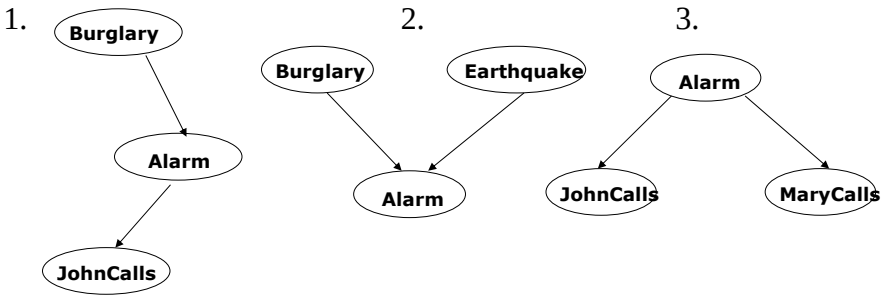
Answer:

- **Chain rule +**
- **Graphical structure** encodes **conditional and marginal independences** among random variables
- **A and B are independent** $P(A, B) = P(A)P(B)$
- **A and B are conditionally independent given C**

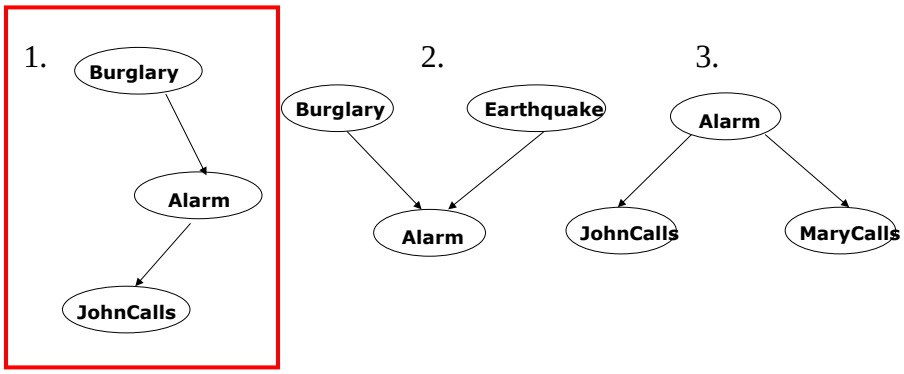
  $P(A \mid C, B) = P(A \mid C)$ $\quad P(A, B \mid C) = P(A \mid C)P(B \mid C)$
- **The graph structure implies the decomposition !!!**

Independences in BBNs

3 basic independence structures:



Independences in BBNs

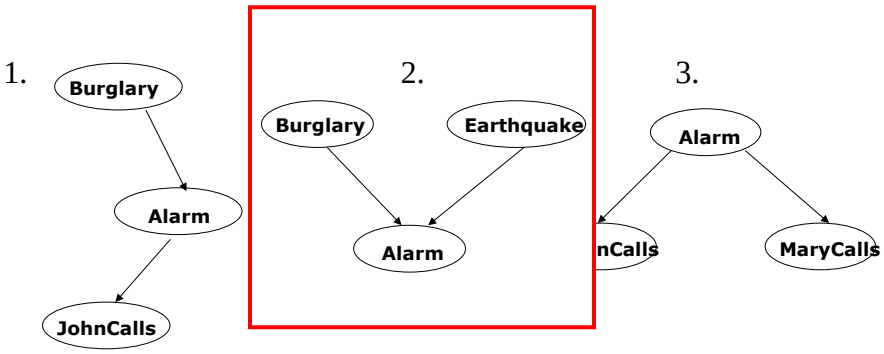


1. JohnCalls **is independent** of Burglary **given** Alarm

$$P(J \mid A, B) = P(J \mid A)$$

$$P(J, B \mid A) = P(J \mid A)P(B \mid A)$$

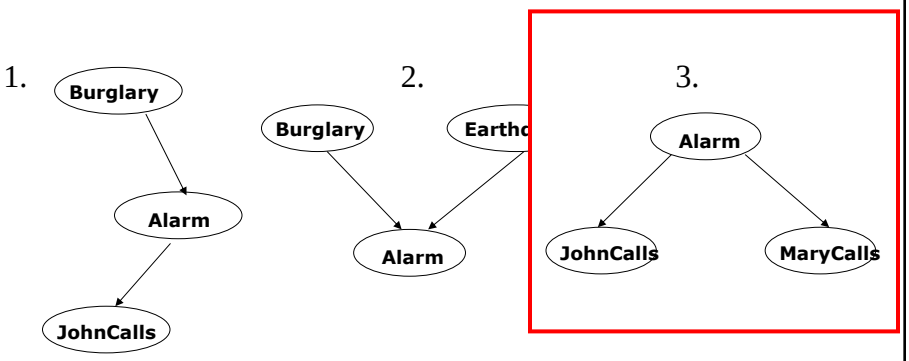
Independences in BBNs



2. Burglary **is independent** of Earthquake (not knowing Alarm)
   Burglary and Earthquake **become dependent** given Alarm !!

$$P(B,E) = P(B)P(E)$$

Independences in BBNs



3. MaryCalls **is independent** of JohnCalls **given** Alarm

$$P(J \mid A, M) = P(J \mid A)$$

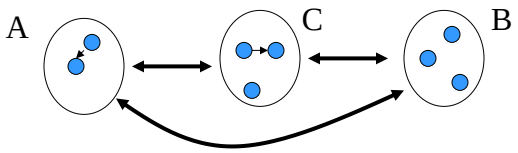
$$P(J, M \mid A) = P(J \mid A)P(M \mid A)$$

Independences in BBN

- BBN distribution models many conditional independence relations among distant variables and sets of variables
- These are defined in terms of the graphical criterion called d-separation
- **D-separation and independence**
  - Let X,Y and Z be three sets of nodes
  - If X and Y are d-separated by Z, then X and Y are conditionally independent given Z
- **D-separation :**
  - A is d-separated from B given C if every undirected path between them is **blocked with C**
- **Path blocking**
  - 3 cases that expand on three basic independence structures

Undirected path blocking

A is d-separated from B given C if every undirected path between them is **blocked**



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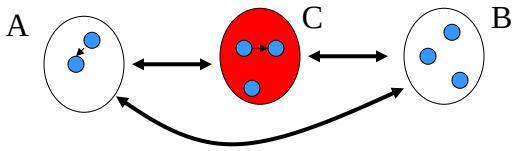

Undirected path blocking

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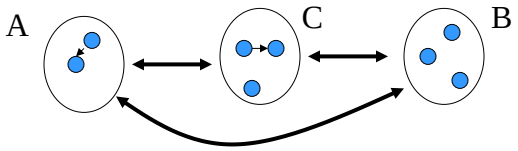
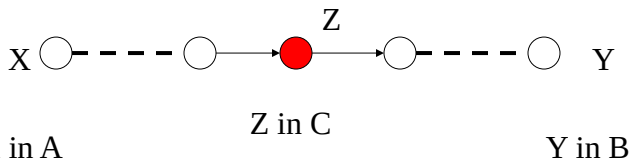


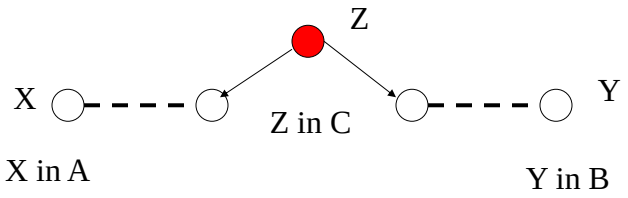
- **1. Path blocking with a linear substructure**



Undirected path blocking

A is d-separated from B given C if every undirected path between them is **blocked**

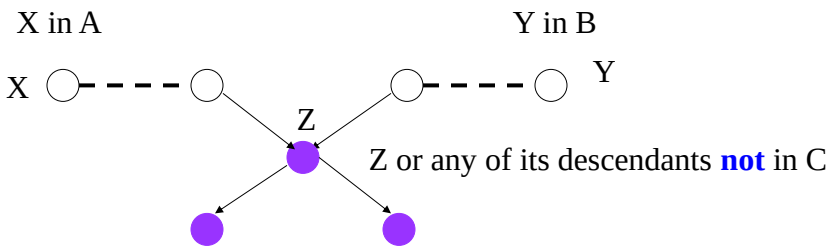
- **2. Path blocking with the wedge substructure**



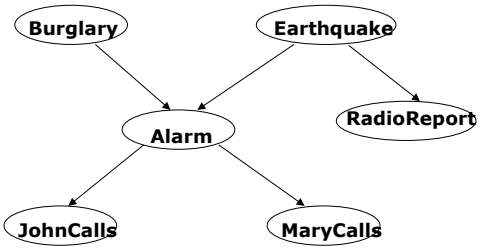
Undirected path blocking

A is d-separated from B given C if every undirected path between them is **blocked**

- **3. Path blocking with the vee substructure**

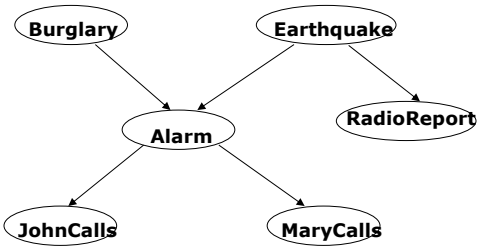


Independences in BBNs



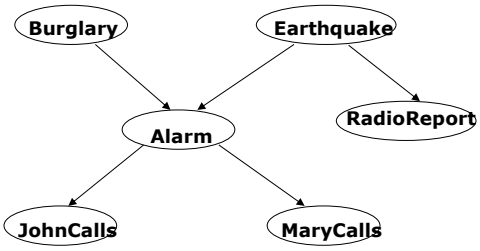
- Earthquake and Burglary are independent given MaryCalls **?**

Independences in BBNs



- Earthquake and Burglary are independent given MaryCalls **F**
- Burglary and MaryCalls are independent (not knowing Alarm) **?**

Independences in BBNs



- Earthquake and Burglary are independent given MaryCalls **F**
- Burglary and MaryCalls are independent (not knowing Alarm) **F**
- Burglary and RadioReport are independent given Earthquake **?**

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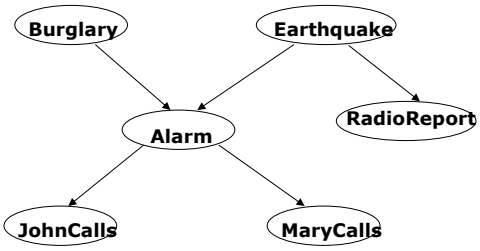
Independences in BBNs



- Earthquake and Burglary are independent given MaryCalls **F**
- Burglary and MaryCalls are independent (not knowing Alarm) **F**
- Burglary and RadioReport are independent given Earthquake **T**
- Burglary and RadioReport are independent given MaryCalls **?**

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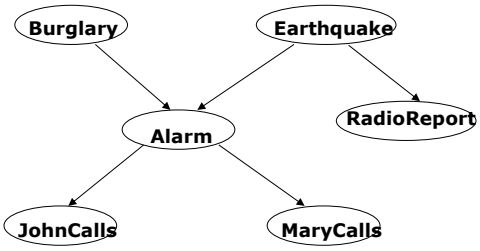
Independences in BBNs



- Earthquake and Burglary are independent given MaryCalls **F**
- Burglary and MaryCalls are independent (not knowing Alarm) **F**
- Burglary and RadioReport are independent given Earthquake **T**
- Burglary and RadioReport are independent given MaryCalls **F**

Bayesian belief networks (BBNs)

Bayesian belief networks

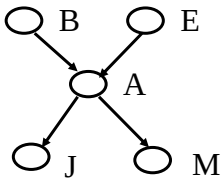
- Represents the full joint distribution over the variables more compactly using the product of local conditionals.
- **So how did we get to local parameterizations?**

$$\mathbf{P}(X_1, X_2, .., X_n) = \prod_{i=1,..n} \mathbf{P}(X_i \mid pa(X_i))$$

- **The decomposition is implied by the set of independences encoded in the belief network.**

Full joint distribution in BBNs

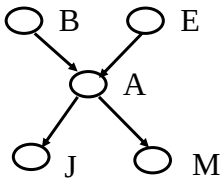
Rewrite the full joint probability using the product rule:



$$P(B=T,E=T,A=T,J=T,M=F)=$$

Full joint distribution in BBNs

Rewrite the full joint probability using the product rule:

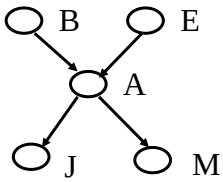


$$P(B=T,E=T,A=T,J=T,M=F)=$$

Product rule

$$=P(J=T\mid B=T,E=T,A=T,M=F)P(B=T,E=T,A=T,M=F)$$

Full joint distribution in BBNs



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Product rule

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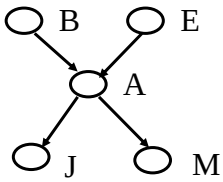
$$=\underline{P(J=T\mid A=T)}P(B=T,E=T,A=T,M=F)$$

Full joint distribution in BBNs



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Product rule

$$P(M=F\mid B=T,E=T,A=T)P(B=T,E=T,A=T)$$

Full joint distribution in BBNs

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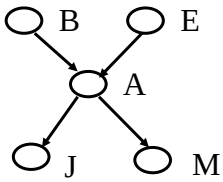

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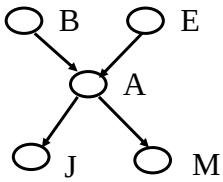

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Full joint distribution in BBNs

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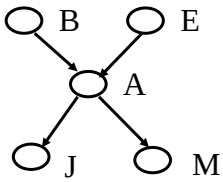

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Full joint distribution in BBNs

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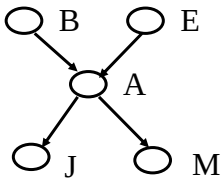

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$$=P(J=T\mid A=T)P(M=F\mid A=T)P(A=T\mid B=T,E=T)P(B=T)P(E=T)$$

Parameter complexity problem

- In the BBN the **full joint distribution** is defined as:

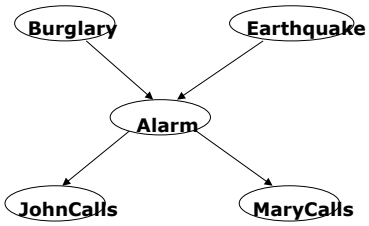
$$\mathbf{P}(X_1, X_2, .., X_n) = \prod_{i=1,..n} \mathbf{P}(X_i \mid pa(X_i))$$

- **What did we save?**

Alarm example: binary (True, False) variables

# of parameters of the full joint:

?



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# of parameters of the full joint:

$$2^5 = 32$$

One parameter is for free:

$$2^5 - 1 = 31$$

# of parameters of the BBN:

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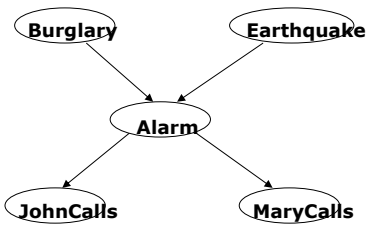
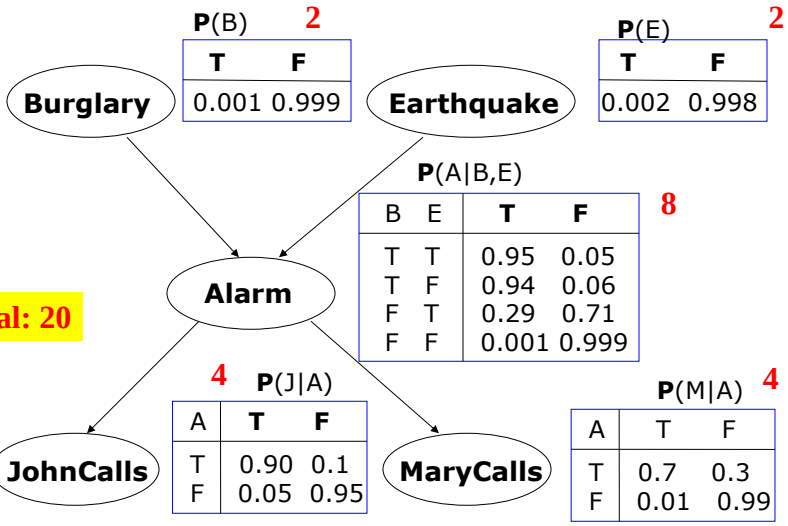


Bayesian belief network: parameters count



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Alarm example: 5 binary (True, False) variables

# of parameters of the full joint:

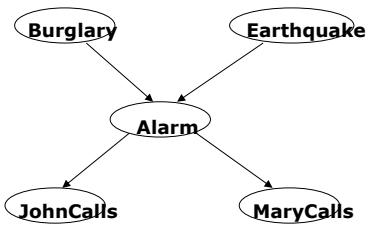
$$2^5 = 32$$

One parameter is for free:

$$2^5 - 1 = 31$$

# of parameters of the BBN:

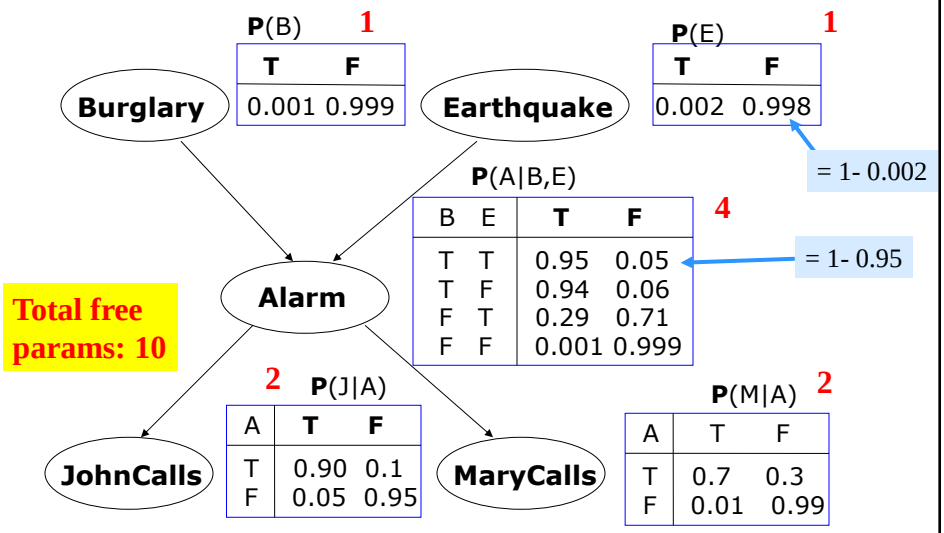
$$2^3 + 2(2^2) + 2(2) = 20$$



One parameter in every conditional is for free:

?

Bayesian belief network: free parameters



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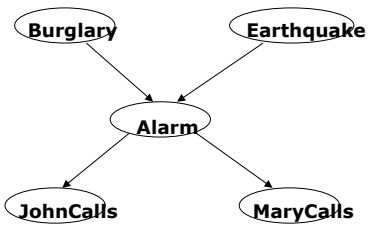

One parameter in every conditional is for free:

$$2^2 + 2(2) + 2(1) = 10$$