Utility theory

Selection based on expected values

- Until now: The optimal action choice was the option that maximized the expected monetary value.
- But is the expected monetary value always the quantity we want to optimize?
Selection based on expected values

• Is the expected monetary value always the quantity we want to optimize?
• Answer: Yes, but only if we are risk-neutral.

• But what if we do not like the risk (we are risk-averse)?
• In that case we may want to get the premium for undertaking the risk (of loosing the money)
• Example: we may prefer to get $101 for sure against $102 in expectation but with the risk of loosing the money
• Problem: How to model decisions and account for the risk?
• Solution: use utility function, and utility theory

Utility function

• Utility function (denoted U)
  – Quantifies how we “value” outcomes, i.e., it reflects our preferences
  – Can be also applied to “value” outcomes other than money and gains (e.g. utility of a patient being healthy, or ill)
• Decision making:
  – uses expected utilities (denoted EU)

\[
EU(X) = \sum_{x \in \Omega_x} P(X = x)U(X = x)
\]

\[
U(X = x) \quad \text{the utility of outcome } x
\]

Important !!!
• Under some conditions on preferences we can always design the utility function that fits our preferences
Utility theory

- Defines axioms on preferences that involve uncertainty and ways to manipulate them.
- Uncertainty is modeled through lotteries
  - **Lottery:**
    
    $[p : A; (1 - p) : C]$  
    
    - Outcome A with probability p  
    - Outcome C with probability (1-p)
- The following six constraints are known as the axioms of utility theory. The axioms are the most obvious semantic constraints on preferences with lotteries.
- **Notation:**
  - $\succ$ - preferable
  - $\sim$ - indifferent (equally preferable)

Axioms of the utility theory

- **Orderability:** Given any two states, the a rational agent prefers one of them, else the two as equally preferable.
  $$ (A \succ B) \lor (B \succ A) \lor (A \sim B) $$
- **Transitivity:** Given any three states, if an agent prefers A to B and prefers B to C, agent must prefer A to C.
  $$ (A \succ B) \land (B \succ C) \Rightarrow (A \succ C) $$
- **Continuity:** If some state B is between A and C in preference, then there is a $p$ for which the rational agent will be indifferent between state B and the lottery in which A comes with probability p, C with probability (1-p).
  $$ (A \succ B \succ C) \Rightarrow \exists p [p : A; (1 - p) : C] \sim B $$
Axioms of the utility theory

- **Substitutability**: If an agent is indifferent between two lotteries, \( A \) and \( B \), then there is a more complex lottery in which \( A \) can be substituted with \( B \).
  
  \[(A \sim B) \Rightarrow [p : A; (1 - p) : C] \sim [p : B; (1 - p) : C]\]

- **Monotonicity**: If an agent prefers \( A \) to \( B \), then the agent must prefer the lottery in which \( A \) occurs with a higher probability.
  
  \[(A \succ B) \Rightarrow (p > q \iff [p : A; (1 - p) : B] \succ [q : A; (1 - q) : B])\]

- **Decomposability**: Compound lotteries can be reduced to simpler lotteries using the laws of probability.
  
  \[
  [p : A ; (1 - p) : [q : B ; (1 - q) : C]] \Rightarrow \\
  [p : A ; (1 - p)q : B ; (1 - p)(1 - q) : C]
  \]

Utility theory

If the agent obeys the axioms of the utility theory, then

1. There exists a real valued function \( U \) such that:

   \[
   U(A) > U(B) \iff A \succ B \\
   U(A) = U(B) \iff A \sim B
   \]

2. The utility of the lottery is the expected utility, that is the sum of utilities of outcomes weighted by their probability.

   \[
   U[p : A.(1 - p) : B] = pU(A) + (1 - p)U(B)
   \]

3. Rational agent makes the decisions in the presence of uncertainty by maximizing its expected utility.
Utility functions

We can design a utility function that fits our preferences if they satisfy the axioms of utility theory.

- But how to design the utility function for monetary values so that they incorporate the risk?
- What is the relation between utility function and monetary values?
  - Assume we lose or gain $1000.
    - Typically this difference is more significant for lower values (around $100 - $1000) than for higher values (~ $1,000,000)
- What is the relation between utilities and monetary value for a typical person?

Utility functions

- What is the relation between utilities and monetary value for a typical person?
- Concave function that flattens at higher monetary values
Utility functions

- Expected utility of a sure outcome of 750 is 750

Assume a lottery L [0.5: 500, 0.5:1000]
- Expected value of the lottery = 750
- Expected utility of the lottery EU(L) is different:
  - EU(L) = 0.5U(500) + 0.5*U(1000)
Utility functions

- Expected utility of the lottery $\text{EU}(\text{lottery L}) < \text{EU}(\text{sure 750})$

- Risk aversion – a bonus is required for undertaking the risk