CS 2750 Machine Learning
Lecture 24

Reinforcement learning

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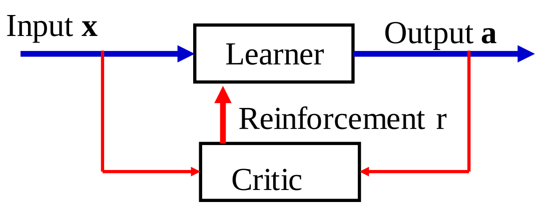
Reinforcement learning

Basics:

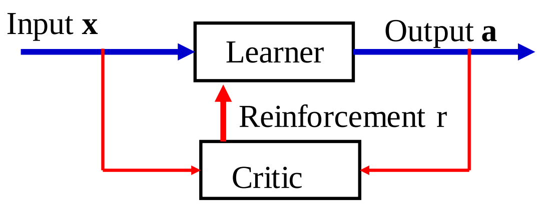


- **Learner interacts with the environment**
  - Receives input with information about the environment (e.g. from sensors)
  - Makes actions that (may) effect the environment
  - Receives a reinforcement signal that provides a feedback on how well it performed

Reinforcement learning

Objective: Learn how to act in the environment in order to maximize the reinforcement signal

- The selection of actions should depend on the input
- A policy $\pi : X \rightarrow A$ maps inputs to actions
- **Goal:** find the optimal policy $\pi : X \rightarrow A$ that gives the best expected reinforcements

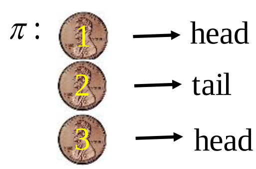


Example: learn how to play games (AlphaGo)

Gambling example

- **Game:** 3 biased coins
  - The coin to be tossed is selected randomly from the three coin options. The agent always sees which coin is going to be played next. The agent makes a bet on either a head or a tail with a wage of \$1. If after the coin toss, the outcome agrees with the bet, the agent wins \$1, otherwise it looses \$1
- **RL model:**
  - **Input:** X – a coin chosen for the next toss,
  - **Action:** A – choice of head or tail the agent bets on,
  - **Reinforcements:** {1, -1}
- **A policy** $\pi : X \rightarrow A$

  Example: $\pi : \left| \begin{array}{l} \textbf{Coin1} \rightarrow \textbf{\textit{head}} \\ \textbf{Coin2} \rightarrow \textbf{\textit{tail}} \\ \textbf{Coin3} \rightarrow \textbf{\textit{head}} \end{array} \right|$



Gambling example

RL model:

- **Input:** X – a coin chosen for the next toss,
- **Action:** A – choice of head or tail the agent bets on,
- **Reinforcements:** {1, -1}
- **A policy**

$$\pi : \left| \begin{array}{l} \textbf{Coin1} \longrightarrow \textbf{\textit{head}} \\ \textbf{Coin2} \longrightarrow \textbf{\textit{tail}} \\ \textbf{Coin3} \quad\ \ \textbf{\textit{head}} \end{array} \right|$$

State, action reward trajectories

	Step0		Step1		Step2		Step k	
state	Coin2		Coin1		Coin2	..	Coin1	
action	Tail		Head		Tail		Head	..
reward	-1		1		1		1	

Gambling example

Learning goal: find the optimal policy

$$\pi^* : X \rightarrow A$$

$$\pi^* : \left| \begin{array}{l} \textbf{Coin1} \longrightarrow \ ? \\ \textbf{Coin2} \longrightarrow \ ? \\ \textbf{Coin3} \longrightarrow \ ? \end{array} \right|$$

π^*:

$\longrightarrow$?
$\longrightarrow$?
$\longrightarrow$?

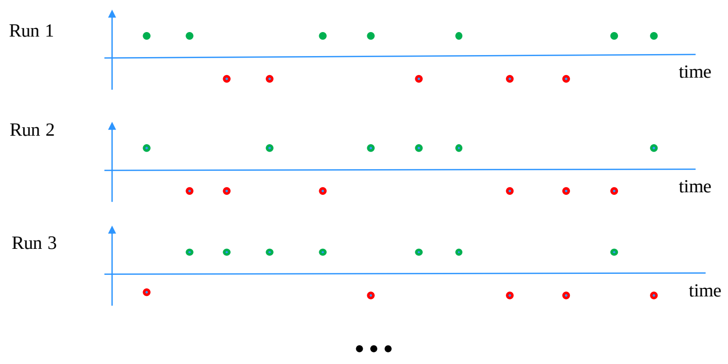
maximizing future expected rewards

$$E(\sum_{t=0}^{T} \gamma^t r_t) \qquad 0 \le \gamma < 1$$

a discount factor = present value of money

Expected rewards

- **Expected rewards for** $\pi : X \rightarrow A$



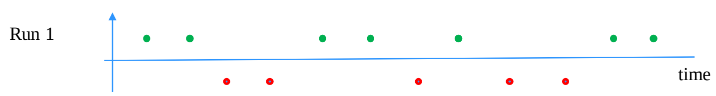
$$E(\sum_{t=0}^{T} r_t)$$

Expectation over many possible reward trajectories defined by $\pi : X \rightarrow A$

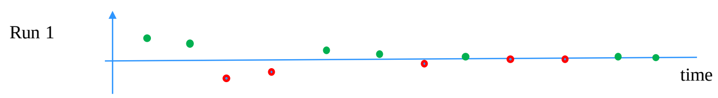
Expected discounted rewards

- **Expected discounting rewards for** $\pi : X \rightarrow A$
- **Discounting with** $0 \leq \gamma < 1$ (future value of money)

No discounting:



Discounting



$$E(\sum_{t=0}^{T} \gamma^t r_t)$$

Expectation over many possible discounted reward trajectories for $\pi : X \rightarrow A$

RL learning: objective functions

- **Objective:**

  Find a policy $\pi^* : X \rightarrow A$

  That maximizes some combination of future reinforcements (rewards) received over time
- **Valuation models** **(quantify how good the mapping is):**
  - **Finite horizon models**

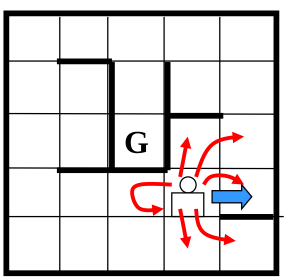
    $E(\sum_{t=0}^{T} r_t)$ Time horizon: $T > 0$

    $E(\sum_{t=0}^{T} \gamma^t r_t)$ Discount factor: $0 \le \gamma < 1$
  - **Infinite horizon discounted model**

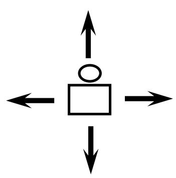
    $E(\sum_{t=0}^{\infty} \gamma^t r_t)$ Discount factor: $0 \le \gamma < 1$
  - **Average reward** $\lim_{T \to \infty} \frac{1}{T} E(\sum_{t=0}^{T} r_t)$

Agent navigation example

- **Agent navigation in the maze**:
  - 4 moves in compass directions
  - Effects of moves are stochastic – we may wind up in other than intended location with a non-zero probability
  - **Objective:** learn how to reach the goal state in the shortest expected time

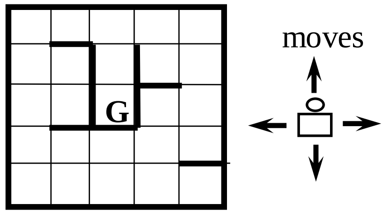


moves



Agent navigation example

- **The RL model:**
  - **Input:** X – a position of an agent
  - **Output:** A –the next move
  - **Reinforcements:** R
    - -1 for each move
    - +100 for reaching the goal
  - **A policy:** $\pi : X \rightarrow A$



$$\pi : \left| \begin{array}{l} \textbf{Position 1} \longrightarrow \textbf{\textit{right}} \\ \textbf{Position 2} \longrightarrow \textbf{\textit{right}} \\ \ldots \\ \textbf{Position 25} \longrightarrow \textbf{\textit{left}} \end{array} \right|$$

- **Goal: find the policy maximizing future expected rewards**

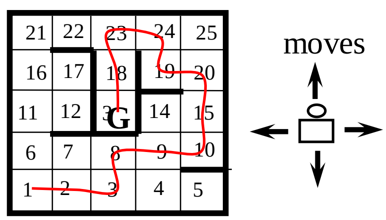
$$E(\sum_{t=0}^{\infty} \gamma^t r_t) \qquad 0 \leq \gamma < 1$$

Agent navigation example

State, action reward trajectories

- **policy**

$$\pi : \left| \begin{array}{l} \textbf{Position 1} \longrightarrow \textbf{\textit{right}} \\ \textbf{Position 2} \longrightarrow \textbf{\textit{right}} \\ \ldots \\ \textbf{Position 25} \longrightarrow \textbf{\textit{left}} \end{array} \right|$$



	Step0	Step1	Step2		Step k	
state	Pos1	Pos2	Pos3	..	Pos15	
action	Right	Right	Up		Up	..
reward	-1	-1	-1		-1	

Effects of actions on the environment

Effect of actions on the environment

– More specifically on the next input **x** to be seen

Case 1. No effect. The distribution over possible **x** is independent of past actions. The rewards received depend only on the current state **x** and the action *a* chosen.

- **Reinforcement learning with immediate rewards**
  - **3 coin example** 

  What coin we see next is not affected by our previous action, hence our action does not effect future rewards

	Step0	Step1	Step2		Step k	
state	Coin2	Coin1	Coin2	..	Coin1	
action	Tail	Head	Tail		Head	..
reward	-1	1	1		1	

Effects of actions on the environment

Effect of actions on the environment

– More specifically on the next input **x** to be seen

Case 2. Actions may effect the environment and next inputs **x**. The distribution of **x** can change due to past actions; the rewards related to the action can be seen with some delay.

- **Learning with delayed rewards** 
  - **Agent navigation example**; a move action effects next position, and hence more distant future rewards

	Step0	Step1	Step2		Step k	
state	Pos1	Pos2	Pos3	..	Pos15	
action	Right	Right	Up		Up	..
reward	-1	-1	-1		-1	

RL with immediate rewards

- **Game:** 3 biased coins
  - The coin to be tossed is selected randomly from the three coin options. The agent always sees which coin is going to be played next. The agent makes a bet on either a head or a tail with a wage of \$1. If after the coin toss, the outcome agrees with the bet, the agent wins \$1, otherwise it looses \$1
- **RL model:**
  - **Input:** X – a coin chosen for the next toss
  - **Action:** A – head or tail the agent bets on
  - **Reinforcements:** {1, -1} (\$1 either won or lost)
- **Learning goal:** **find the optimal policy** $\pi^*: X \rightarrow A$

  maximizing the future expected profits over time

  $E(\sum_{t=0}^{\infty} \gamma^t r_t)$ $\qquad 0 \le \gamma < 1$ $\qquad$ a discount factor

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RL with immediate rewards

- **Expected reward** $\qquad E(\sum_{t=0}^{\infty} \gamma^t r_t) \qquad 0 \le \gamma < 1$
- **Immediate reward case:**
  - Reward depends only on **x** and the action choice
  - The action does not affect the environment and hence future inputs (states) and future rewards:

General Trajectory	Step0		Step1		Step2			Step k	
state	x_0	X	x_1	X	x_2	X	X	x_k	
action	a_0		a_1		a_2		..	a_k	..
Reward	$r_{x0,a0}$		$r_{x1,a1}$		$r_{x2,a2}$			$r_{xk,ak}$	

$$E(\sum_{t=0}^{\infty} \gamma^t r_t) = E(r_{x0,a0}) + E(\gamma r_{x1,a1}) + E(\gamma^2 r_{x2,a2}) + \ldots E(\gamma^k r_{xk,ak}) + \ldots$$
$$= E(r_{x0,a0}) + \gamma E(r_{x1,a1}) + \gamma^2 E(r_{x2,a2}) + \ldots \gamma^k E(r_{xk,ak}) + \ldots$$

RL with immediate rewards

Immediate reward case:

- Reward for input **x** and the action choice *a* may vary
- **Expected one-step reward for the input x and action *a*:**

$$R(\mathbf{x},a) = E(r_{\mathbf{x},a})$$

  – For the coin bet problem it is:

$$R(\mathbf{x},a_i) = \sum_j r(\omega_j \mid a_i, \mathbf{x}) P(\omega_j \mid \mathbf{x}, a_i)$$

ω_j : an outcome of the coin toss x

$r(\omega_j \mid a_i, \mathbf{x})$: reward for an outcome and the bet made on **x**

- **Expected one step reward for a policy** $\pi : X \rightarrow A$

$$R(\mathbf{x}, \pi(x)) = E(r_{\mathbf{x},\pi(x)})$$

RL with immediate rewards

- **Expected reward**

$$E(\sum_{t=0}^{\infty} \gamma^t r_t) = E(r_{x0,a0}) + E(\gamma r_{x1,a1}) + E(\gamma^2 r_{x2,a2}) + \ldots E(\gamma^k r_{xk,ak}) + \ldots$$
$$= E(r_{x0,a0}) + \gamma E(r_{x1,a1}) + \gamma^2 E(r_{x2,a2}) + \ldots \gamma^k E(r_{xk,ak}) + \ldots$$

- **Optimizing the expected reward :**

$$\max E(\sum_{t=0}^{\infty} \gamma^t r_t) = \max E(r_{x0,a0}) + \max E(\gamma r_{x1,a1}) + \ldots \max E(\gamma^k r_{xk,ak}) + \ldots$$
$$= \max E(r_{x0,a0}) + \gamma \max E(r_{x1,a1}) + \ldots \gamma^k \max E(r_{xk,ak}) + \ldots$$
$$= \max_{a_0} R(x_0,a_0) + \gamma \max_{a_1} R(x_1,a_1) + \ldots \gamma^k \max_{a_k} R(x_k,a_k) + \ldots$$

- **Optimal strategy:** $\pi^* : X \rightarrow A$

$$\pi^*(\mathbf{x}) = \arg\max_a R(\mathbf{x},a)$$

RL with immediate rewards

The optimal choice assumes we know the expected reward $R(\mathbf{x}, a)$

- **Then:** $\pi^*(\mathbf{x}) = \arg\max_a R(\mathbf{x}, a)$

Caveats

- **We do not know the expected reward** $R(\mathbf{x}, a)$
  - We need to estimate it using $\tilde{R}(\mathbf{x}, a)$ from interaction
- **We cannot determine the optimal policy if the estimate of the expected reward is not good**
  - We need to try also actions that look suboptimal wrt the current estimates of $\tilde{R}(\mathbf{x}, a)$

RL with immediate rewards

- **Problem:** In the RL framework we do not know $R(\mathbf{x}, a)$
  - The expected reward for performing action *a* at input **x**
- **Solution:**
  - For each input **x** try different actions *a*
  - Estimate $R(\mathbf{x}, a)$ using the average of observed rewards

$$\boxed{\tilde{R}(\mathbf{x}, a) = \frac{1}{N_{x,a}} \sum_{i=1}^{N_{x,a}} r_i^{x,a}}$$

  - Action choice $\pi(\mathbf{x}) = \arg\max_a \tilde{R}(\mathbf{x}, a)$
  - Accuracy of the estimate: statistics (Hoeffding's bound)

$$P\left(\left|\tilde{R}(\mathbf{x}, a) - R(\mathbf{x}, a)\right| \geq \varepsilon\right) \leq \exp\left[-\frac{2\varepsilon^2 N_{x,a}}{(r_{\max} - r_{\min})^2}\right] \leq \delta$$

  - Number of samples: $N_{x,a} \geq \frac{(r_{\max} - r_{\min})^2}{2\varepsilon^2} \ln \frac{1}{\delta}$

RL with immediate rewards

- **On-line (stochastic approximation)**
  - An alternative way to estimate $R(\mathbf{x}, a)$
- **Idea:**
  - choose action *a* for input **x** and observe a reward $r^{x,a}$
  - Update an estimate in every step *i*

$$\boxed{\tilde{R}(\mathbf{x}, a)^{(i)} \leftarrow (1-\alpha(i))\tilde{R}(\mathbf{x}, a)^{(i-1)} + \alpha(i)\, r_i^{x,a}}$$ $\alpha(i)$ - a learning rate

- **Convergence property:** The approximation converges in the limit for an appropriate learning rate schedule.
- Assume: $\alpha(n(x,a))$ - is a learning rate for *n*th trial of (*x*,*a*) pair
- Then the converge is assured if:

1. $\sum_{i=1}^{\infty} \alpha(i) = \infty$ 2. $\sum_{i=1}^{\infty} \alpha(i)^2 < \infty$

RL with immediate rewards

- At any step in time *i* during the experiment we have estimates of expected rewards for each (*coin, action*) pair:

$\tilde{R}(coin1, head)^{(i)}$
$\tilde{R}(coin1, tail)^{(i)}$
$\tilde{R}(coin2, head)^{(i)}$
$\tilde{R}(coin2, tail)^{(i)}$
$\tilde{R}(coin3, head)^{(i)}$
$\tilde{R}(coin3, tail)^{(i)}$

- Assume the next coin to play in step (*i*+1) is coin 2 and we pick head as our bet. Then we update $\tilde{R}(coin2, head)^{(i+1)}$ using the observed reward and one of the update strategy above, and keep the reward estimates for the remaining (coin, action) pairs unchanged, e.g. $\tilde{R}(coin2, tail)^{(i+1)} = \tilde{R}(coin2, tail)^{(i)}$

Exploration vs. Exploitation in RL

The (learner) actively interacts with the environment via actions:

- At the beginning the learner does not know anything about the environment
- It gradually gains the experience and learns how to react to the environment

Dilemma (exploration-exploitation):

- After some number of steps, should I select the best current choice (**exploitation**) or try to learn more about the environment (**exploration**)?
- **Exploitation** may involve the selection of a sub-optimal action and prevent the learning of the optimal choice
- **Exploration** may spend to much time on trying bad currently suboptimal actions

Exploration vs. Exploitation

- In the RL framework
  - the (learner) actively interacts with the environment and **choses the action** to play for the current input $\mathbf{x}$
  - Also at any point in time it has an estimate of $\tilde{R}(\mathbf{x}, a)$ for any (*input,action*) pair
- **Dilemma for choosing the action to play for x:**
  - Should the learner choose the current best choice of action (exploitation)
    $$\hat{\pi}(\mathbf{x}) = \arg\max_{a \in A} \tilde{R}(\mathbf{x}, a)$$
  - Or choose some other action a which may help to improve its $\tilde{R}(\mathbf{x}, a)$ estimate (exploration)

  This dilemma is called **exploration/exploitation dilemma**
- **Different exploration/exploitation strategies exist**

Exploration vs. Exploitation

- **Uniform exploration:**
  - Uses exploration parameter $0 \le \varepsilon \le 1$
  - Choose the "current" best choice with probability $1-\varepsilon$

$$\hat{\pi}(\mathbf{x}) = \arg\max_{a \in A} \widetilde{R}(\mathbf{x}, a)$$

  - All other choices are selected with a uniform probability $\dfrac{\varepsilon}{|A|-1}$

Advantages:

- Simple, easy to implement

Disadvantages:

- Exploration more appropriate at the beginning when we do not have good estimates of $\widetilde{R}(\mathbf{x}, a)$
- Exploitation more appropriate later when we have good estimates

Exploration vs. Exploitation

- **Boltzman exploration**
  - The action is chosen randomly but proportionally to its current expected reward estimate
  - Can be tuned with a temperature parameter T to promote exploration or exploitation
- Probability of choosing action *a*

$$p(a \mid \mathbf{x}) = \frac{\exp\left[\widetilde{R}(x,a)/T\right]}{\sum_{a' \in A} \exp\left[\widetilde{R}(x,a')/T\right]}$$

- **Effect of T:**
  - For high values of T, $p(a \mid x)$ is uniformly distributed for all actions
  - For low values of T, $p(a \mid x)$ of the action with the highest value of $\widetilde{R}(\mathbf{x}, a)$ is approaching 1