Solutions to problem set 3

Problem 1. Constraint satisfaction and Constraint Propagation

The CSP problem is represented by the following graph.

Let’s summarize the assignments and constraints we have.

<table>
<thead>
<tr>
<th>Equations:</th>
<th>Constraints:</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x = 2$</td>
<td>$x \equiv z \pmod{3}$</td>
</tr>
<tr>
<td>(init. assgn.) $y = 0$</td>
<td>$y \equiv z \pmod{5}$</td>
</tr>
<tr>
<td>$t = 0$</td>
<td>$z \equiv w \pmod{4}$</td>
</tr>
</tbody>
</table>

Moreover, there are the constraints that $0 \leq x, y, z, w, u, v \leq 9$.

*Equations* are the formulas of the form *variable = value*. Disequations are the formulas of the form *var ≠ value.*
Forward checking

<table>
<thead>
<tr>
<th>Equations:</th>
<th>x = 2</th>
<th>y = 0</th>
<th>v = 0</th>
<th>t = 0</th>
</tr>
</thead>
<tbody>
<tr>
<td>Disequations:</td>
<td>none</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Forward checking infers: (a) disequations from equations and constraints, and (b) equations through the exhaustion of alternatives. Intuitively, the process of inferring a disequation corresponds to checking whether the new assignment of value to a variable (equation) restricts domains of variables connected to it in the constraint graph. These restrictions are represented as disequations (invalid assignments).

As an example, consider the assignment \( x = 2 \). According to the mod constraints, we can delete values 0,1,3,4,6,7,9 from the domain of \( z \), to ensure the consistency. This is equivalent to inferring disequations \( z \neq 0, z \neq 1, z \neq 3, \ldots, z \neq 8 \). So now, \( \text{dom}(z) = \{2, 5, 9\} \).

<table>
<thead>
<tr>
<th>Equations:</th>
<th>x = 2</th>
<th>y = 0</th>
<th>v = 0</th>
<th>t = 0</th>
</tr>
</thead>
<tbody>
<tr>
<td>Disequations:</td>
<td>( z \neq 0, 1, 3, 4, 6, 7, 8 )</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Now, assume equation \( y = 0 \). Forward checking along the second constraint forces us to delete 2 and 9 from \( \text{dom}(z) \). Again, this means we have inferred the disequations \( z \neq 2, z \neq 9 \). Now, \( \text{dom}(z) = \{5\} \) and we can infer that \( z = 5 \) – by exhaustion of alternatives.

<table>
<thead>
<tr>
<th>Equations:</th>
<th>x = 2</th>
<th>y = 0</th>
<th>v = 0</th>
<th>t = 0</th>
<th>z = 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Disequations:</td>
<td>( z \neq 0, 1, 2, 3, 4, 6, 7, 8, 9 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

So far we have used equations \( x = 2, y = 0 \). We have two more equations we haven’t evaluated yet, \( v = 0 \) and \( z = 5 \). By taking \( v = 0 \) we can delete 1,2,4,5,7,8 from the domain of \( u \) via forward checking, which gives:

<table>
<thead>
<tr>
<th>Equations:</th>
<th>x = 2</th>
<th>y = 0</th>
<th>v = 0</th>
<th>t = 0</th>
<th>z = 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Disequations:</td>
<td>( z \neq 0, \ldots z \neq 9 ) (except 5)</td>
<td>( u \neq 1, 2, 4, 5, 7, 8 )</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
So, \( \text{dom}(u) = \{0, 3, 6, 9\} \).

The equation \( z = 5 \) (inferred) triggers further constraint propagation. In particular we delete values 2, 3, 4, 6, 7, 8 of \( w \) as inconsistent with \( z = 5 \), or equivalently infer that \( \text{dom}(w) = \{1, 5, 9\} \). The final state of the forward checking procedure is:

\[
\begin{array}{c|c|c}
\text{Equations:} & \text{Disequations:} \\
\hline
x = 2 & z \neq 0, \ldots z \neq 9 \text{ (except 5)} \\
y = 0 & u \neq 1, 2, 4, 5, 7, 8 \\
v = 0 & w \neq 0, 2, 3, 4, 6, 7, 8 \\
t = 0 & \\
z = 5 & \\
w = 9 & \\
\end{array}
\]

and now no further conclusions can be made.

**Arc consistency**

Observe that all inferences made in FC are inferences that would be also made by the arc consistency method. In addition, arc consistency procedure also checks the consistency of remaining values for pairs of variables along the arc that connects them. Let us consider variables \( w, u \) and arc \( w \rightarrow u \).

The remaining values for \( w \) are \( \text{dom}(w) = \{1, 5, 9\} \) and for \( u \) these are \( \text{dom}(u) = \{0, 3, 6, 9\} \). Let us check if individual values of \( w \) are consistent with the remaining values for \( u \). Let us first assume \( w = 1 \). Values of \( u \) that are consistent with \( w = 1 \) across mod 3 constraint are \{1, 4, 7\}. Unfortunately none of these values is in the current \( \text{dom}(u) = \{0, 3, 6, 9\} \), hence the value \( w = 1 \) is inconsistent with \( u \) and we can infer \( w \neq 1 \). Now assume \( w = 5 \). This value is consistent with values of \{2, 5, 8\} for \( u \). Once again, none of these values is consistent with current \( \text{dom}(u) = \{0, 3, 6, 9\} \), hence \( w \neq 5 \) can be inferred. Finally, let us try \( w = 9 \). The value is consistent with values 3, 6, 9. These values indeed overlap with the remaining values \( \text{dom}(u) = \{0, 3, 6, 9\} \) hence the assignment is consisted. Taking the new disequations \( w \neq 1 \) and \( w \neq 5 \) together with the previously derived disequations, we can infer \( w = 9 \) through the exhaustion of alternatives. Now, all arcs are consistent (verify!) and the resulting partial assignment is:

\[
\begin{array}{c|c|c}
\text{Equations:} & \text{Disequations:} \\
\hline
x = 2 & z \neq 0, \ldots z \neq 9 \text{ (except 5)} \\
y = 0 & u \neq 1, 2, 4, 5, 7, 8 \\
v = 0 & w \neq 0, 2, 3, 4, 5, 6, 7, 8 \\
t = 0 & \\
z = 5 & \\
w = 9 & \\
\end{array}
\]
The only variable that remains unassigned is $u$. It is only partly constrained ($\text{dom}(u) = \{0, 3, 6, 9\}$) and no new equation or disequation can be inferred.