## CS 1571 Introduction to AI
## Lecture 13

# Propositional logic

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## Midterm

- **The midterm** for the course will be held on
  - October 28, 2014
  - In class exam
  - Closed book
  - Material covered by October 23, 2014

## Logical inference problem

**Logical inference problem:**

- **Given:**
  - a knowledge base KB (a set of sentences) and
  - a sentence $\alpha$ (called **a theorem)**,
- **Does a KB semantically entail** $\alpha$ ? $KB \models \alpha$ ?

**In other words:**

- In all interpretations in which sentences in the KB are true, is $\alpha$ also true?



## Logical inference problem

**Logical inference problem:**

- **Given:**
  - a knowledge base KB (a set of sentences) and
  - a sentence $\alpha$ (called **a theorem)**,
- **Does a KB semantically entail** $\alpha$ ? $KB \models \alpha$

**Approaches to solve the logical inference problem:**

- **Truth-table approach**
- **Inference rules**
- **Conversion to SAT**
  - **Resolution refutation**

## Properties of inference solutions

- **Truth-table approach**
  - **Blind**
  - **Exponential in the number of variables**
- **Inference rules**
  - **More efficient**
  - **Many inference rules to cover logic**
- **Conversion to SAT - Resolution refutation**
  - **More efficient**
  - **Sentences must be converted into CNF**
  - **One rule – the resolution rule - is sufficient to perform all inferences**



## KB in restricted forms

If the sentences in the KB are restricted to some special forms some of the sound inference rules may become complete

**Example:**

- **Horn form (Horn normal form)**
  - a clause with **at most one positive literal**

$$(A \vee \neg B) \wedge (\neg A \vee \neg C \vee D)$$

Can be written also as:

$$(B \Rightarrow A) \wedge ((A \wedge C) \Rightarrow D)$$

- **Two inference rules that are sound and complete for KBs in the Horn normal form:**
  - **Resolution**
  - **Modus ponens**

## KB in Horn form

- **Horn form:** a clause with **at most one positive literal**

$$(A \vee \neg B) \wedge (\neg A \vee \neg C \vee D)$$

- **Not all sentences in propositional logic can be converted into the Horn form**
- **KB in Horn normal form**:
  - Two types of propositional statements:
    - **Rules** $(\neg B_1 \vee \neg B_2 \vee \dots \neg B_k \vee A)$

$$\equiv$$

$$(\neg(B_1 \wedge B_2 \wedge \dots B_k) \vee A)$$

$$\equiv$$

$$(B_1 \wedge B_2 \wedge \dots B_k \Rightarrow A)$$

    - Propositional symbols: **facts** $B$



## Why KB in Horn form is useful?

**KB in Horn normal form**:

- **Rules (in implicative form): If then statements known to be true**

$$(B_1 \wedge B_2 \wedge \dots B_k \Rightarrow A)$$

**If** 1. The stain of the organism is gram-positive, and
2. The morphology of the organism is coccus, and
3. The growth conformation of the organism is chains
**Then** the identity of the organism is streptococcus

- **Facts = propositions known to be true**

Examples: $B_1$ or

The stain of the organism is gram-positive

**Inferences:** let us infer new true propositions, such as $A$, **or** the identity of the organism is streptococcus in the rule conclusion

These are referred as inferences on propositional symbols

## KB in Horn form

- **Application of the resolution rule:**
  - Infers new facts from previous facts

$$\frac{(A \vee \neg B),\ B}{A} \qquad \frac{(A \vee \neg B), \quad (B \vee \neg C)}{(A \vee C)}$$

  - Resolution is **sound and complete** for inferences on propositional symbols for KB in the Horn normal form (clausal form)

- Similarly, **modus ponens is sound and complete** when the HNF is written in the implicative form



## Complexity of inferences for KBs in HNF

**Question:**
**How efficient the inferences in the HNF can be?**
**Answer:**
**Inference on propositional symbols →**
**Procedures linear in the size of the KB in the Horn form exist.**

- Size of a clause: the number of literals it contains.
- Size of the KB in the HNF: the sum of the sizes of its elements.

**Example:**

$$A, B, (A \wedge B \Rightarrow C), (C \Rightarrow D), (C \Rightarrow E), (E \wedge F \Rightarrow G)$$

or

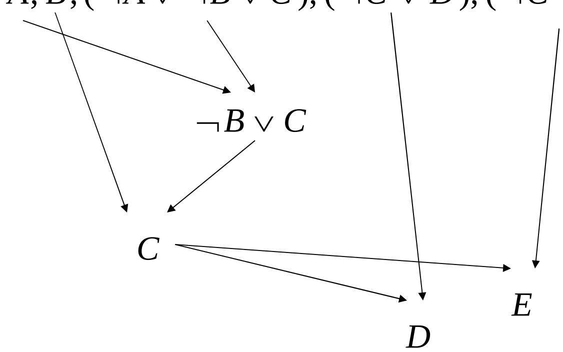
$$A, B, (\neg A \vee \neg B \vee C), (\neg C \vee D), (\neg C \vee E), (\neg E \vee \neg F \vee G)$$

The size is: 12

## Complexity of inferences for KBs in HNF

How to do the inference on propositional symbols? If the HNF (is in the clausal form) we can apply resolution.

$$A, B, (\neg A \vee \neg B \vee C), (\neg C \vee D), (\neg C \vee E), (\neg E \vee \neg F \vee G)$$



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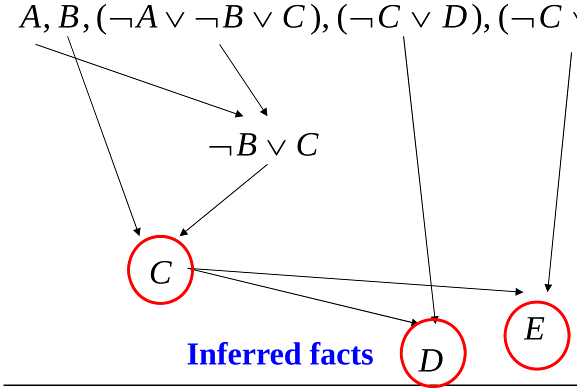
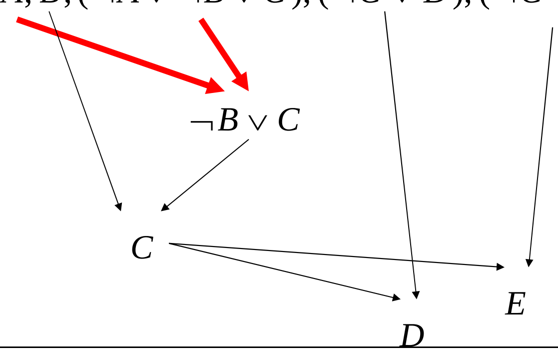


## Complexity of inferences for KBs in HNF

**Features:**

- Every resolution is a **positive unit resolution**; that is, a resolution in which **one clause is a positive unit clause** (i.e., a proposition symbol).

$$A, B, (\neg A \vee \neg B \vee C), (\neg C \vee D), (\neg C \vee E), (\neg E \vee \neg F \vee G)$$






## Complexity of inferences for KBs in HNF

**Features:**

- At each resolution, the input clause which is not a unit clause is a logical consequence of the result of the resolution. (Thus, the input clause may be deleted upon completion of the resolution operation.)

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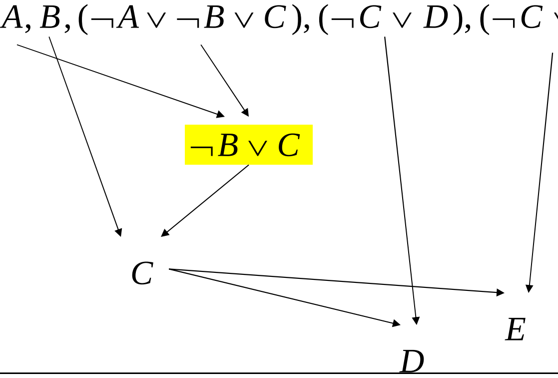

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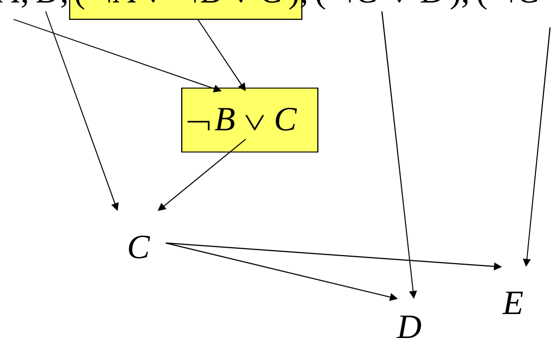


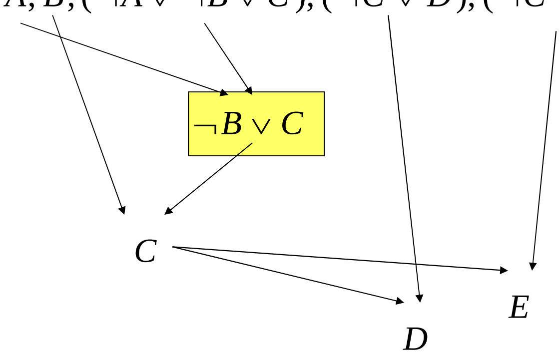



# Complexity of inferences for KBs in HNF

**Features:**

- Following this deletion, the size of the KB (the sum of the lengths of the remaining clauses) is one less than it was before the operation.)

$$A, B, ~~(\neg A \vee \neg B \vee C)~~, (\neg C \vee D), (\neg C \vee E), (\neg E \vee \neg F \vee G)$$

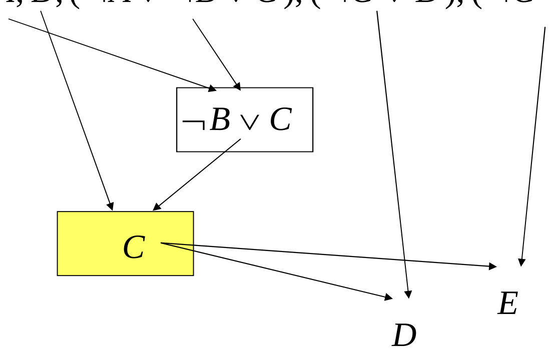


## Complexity of inferences for KBs in HNF

**Features:**

- Following the deletion, the size of the KB (the sum of the lengths of the remaining clauses) is one less than it was before the operation.). **Now let us see one more step …**

$A, B,$ ~~$(\neg A \vee \neg B \vee C)$~~$, (\neg C \vee D), (\neg C \vee E), (\neg E \vee \neg F \vee G)$



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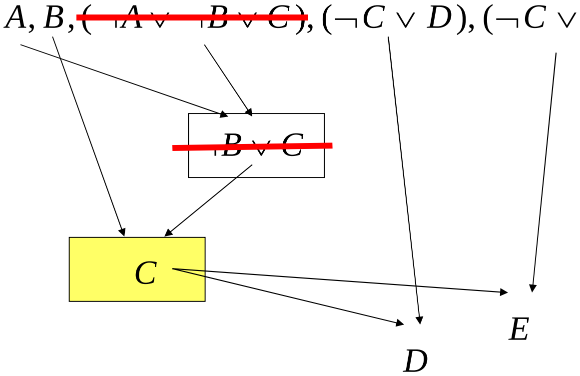
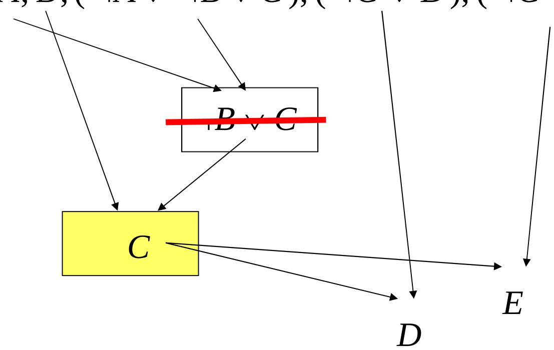


## Complexity of inferences for KBs in HNF

**Features:**

- If n is the size of the KB, then at most n positive unit resolutions may be performed on it.

$A, B, \sout{(\neg A \vee \neg B \vee C)}, (\neg C \vee D), (\neg C \vee E), (\neg E \vee \neg F \vee G)$






## Complexity of inferences for KBs in HNF

**A linear time resolution algorithm:**

- **The number of positive unit resolutions is limited to the size of the formula (n)**
- **But to assure overall linear time we need to access each proposition in a constant time:**
- Data structures indexed by proposition names may be accessed in constant time. (This is possible if the proposition names are number in a range (e.g., 1..n), so that array lookup is the access operation.
- If propositions are accessed by name, then a symbol table is necessary, and the algorithm will run in time O(n · log(n)).

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## Forward and backward chaining

Two inference procedures based on **modus ponens** for **Horn KBs**:

- **Forward chaining**

  **Idea:** Whenever the premises of a rule are satisfied, infer the conclusion. Continue with rules that became satisfied.

- **Backward chaining (goal reduction)**

  **Idea:** To prove the fact that appears in the conclusion of a rule prove the premises of the rule. Continue recursively.

Both procedures are **complete for KBs in the Horn form** !!!



## Forward chaining example

- **Forward chaining**

  **Idea:** Whenever the premises of a rule are satisfied, infer the conclusion. Continue with rules that became satisfied.

Assume the KB with the following rules and facts:

KB: R1: $A \wedge B \Rightarrow C$

R2: $C \wedge D \Rightarrow E$

R3: $C \wedge F \Rightarrow G$

---

F1: $A$

F2: $B$

F3: $D$

Theorem: $E$ ?

## Forward chaining example

**Theorem:** $E$

KB: R1: $A \wedge B \Rightarrow C$

R2: $C \wedge D \Rightarrow E$

R3: $C \wedge F \Rightarrow G$

---

F1: $A$
F2: $B$
F3: $D$



## Forward chaining example

**Theorem:** $E$

KB: R1: $A \wedge B \Rightarrow C$

R2: $C \wedge D \Rightarrow E$

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F1: $A$
F2: $B$
F3: $D$

**Rule R1 is satisfied.**

F4: $C$

## Forward chaining example

**Theorem:** $E$

KB: R1: $A \wedge B \Rightarrow C$

R2: $C \wedge D \Rightarrow E$

R3: $C \wedge F \Rightarrow G$

---

F1: $A$
F2: $B$
F3: $D$

**Rule R1 is satisfied.**

F4: $C$

**Rule R2 is satisfied.**

F5: $E$ ✓



## Forward chaining

- Efficient implementation: linear in the size of the KB
- **Example:**

$P \Rightarrow Q$

$L \wedge M \Rightarrow P$

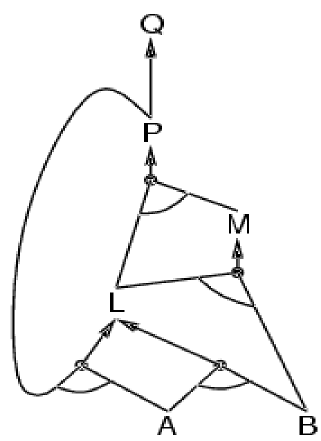
$B \wedge L \Rightarrow M$

$A \wedge P \Rightarrow L$

$A \wedge B \Rightarrow L$

$A$

$B$



## Forward chaining

- Count the number of facts in the antecedent of the rule

$P \Rightarrow Q$

$L \wedge M \Rightarrow P$

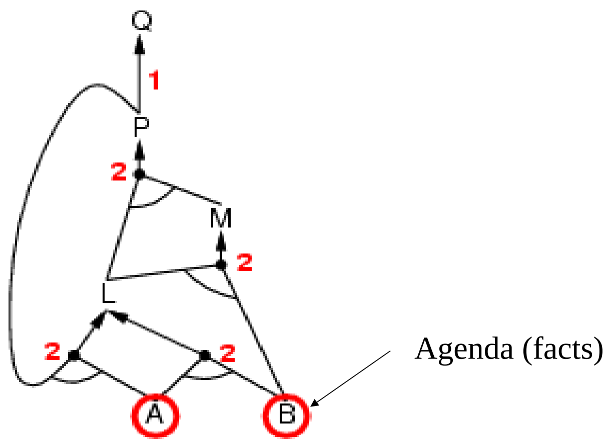
$B \wedge L \Rightarrow M$

$A \wedge P \Rightarrow L$

$A \wedge B \Rightarrow L$

$A$

$B$






## Forward chaining

- Inferred facts decrease the count

$P \Rightarrow Q$

$L \wedge M \Rightarrow P$

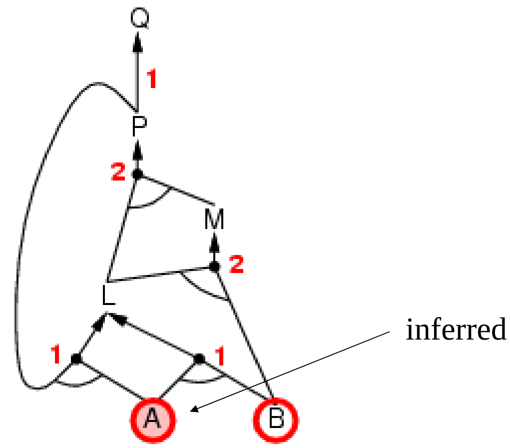
$B \wedge L \Rightarrow M$

$A \wedge P \Rightarrow L$

$A \wedge B \Rightarrow L$

$A$

$B$



## Forward chaining

- New facts can be inferred when the count associated with a rule becomes 0

$P \Rightarrow Q$

$L \wedge M \Rightarrow P$

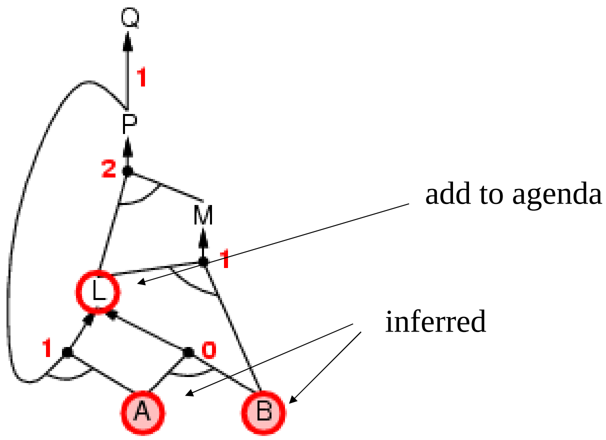
$B \wedge L \Rightarrow M$

$A \wedge P \Rightarrow L$

$A \wedge B \Rightarrow L$

$A$

$B$






## Forward chaining

- 

$P \Rightarrow Q$

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$A$

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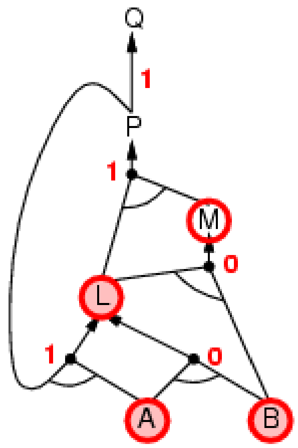

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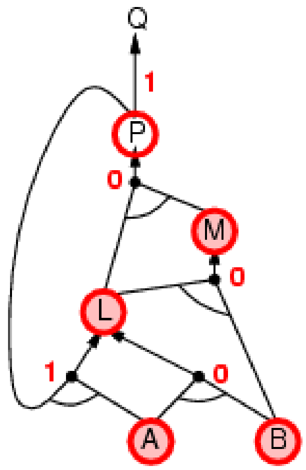




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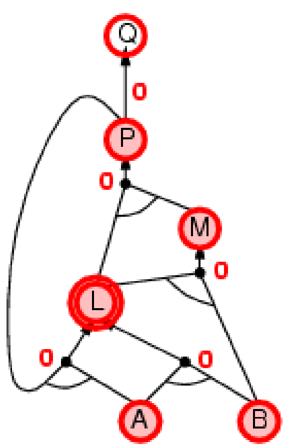

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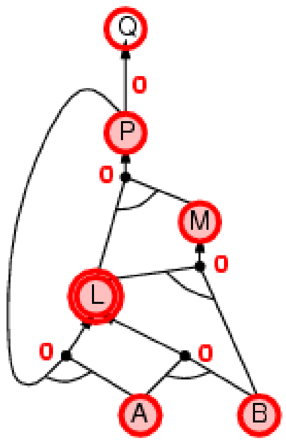
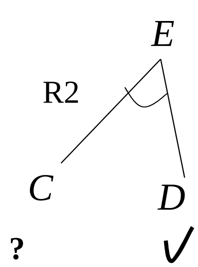





## Backward chaining example

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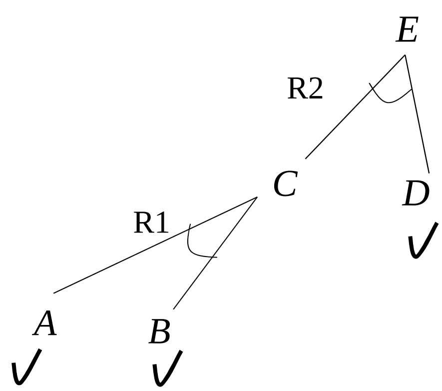
- Backward chaining is more focused:
  - tries to prove the theorem only

# Backward chaining example

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# Backward chaining

- **Efficient implementation**

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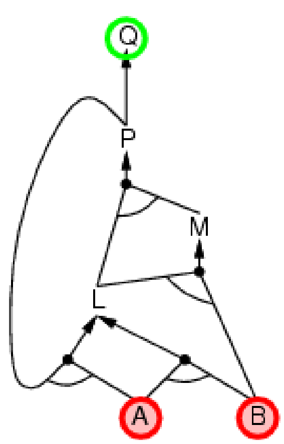

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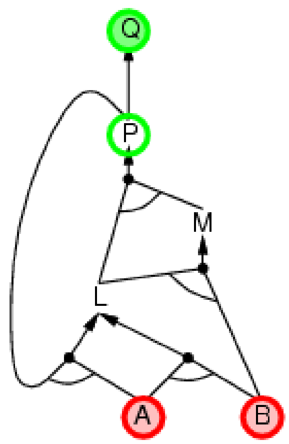




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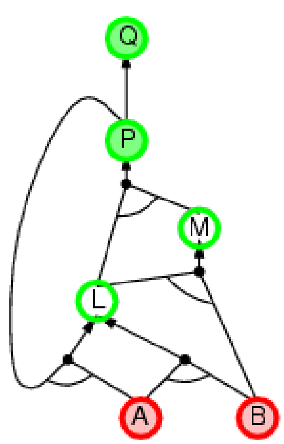

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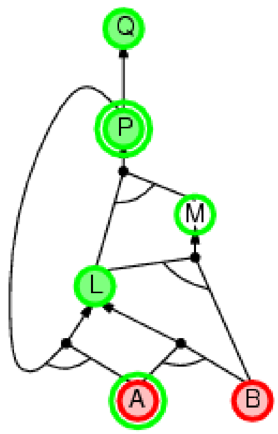




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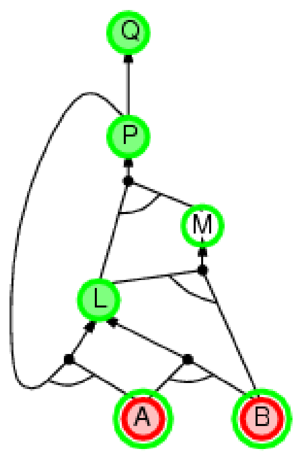

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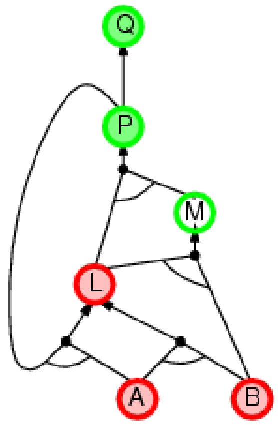




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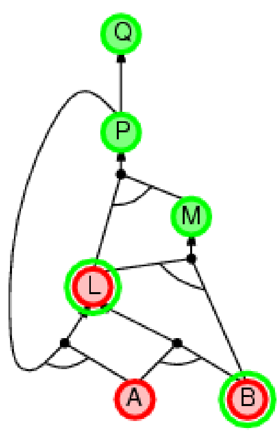

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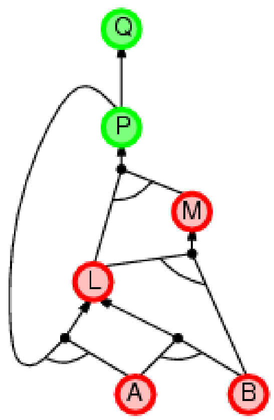


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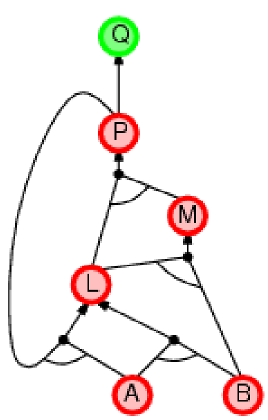


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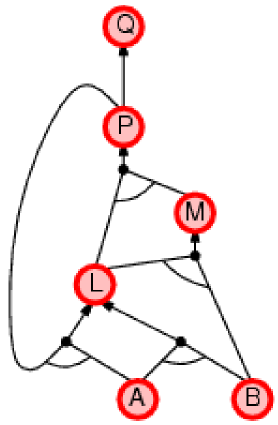




# Forward vs Backward chaining

- **FC is data-driven**, automatic, unconscious processing,
  - e.g., object recognition, routine decisions
- May do lots of work that is irrelevant to the goal
- **BC is goal-driven**, appropriate for problem-solving,
  - e.g., Where are my keys? How do I get into a PhD program?
- Complexity of BC can be **much less** than **linear in size of KB**

## KB agents based on propositional logic

- Propositional logic allows us to build **knowledge-based agents** capable of answering queries about the world by inferring new facts from the known ones
- **Example:** an agent for diagnosis of a bacterial disease

**Facts:** The stain of the organism is gram-positive
The growth conformation of the organism is chains

**Rules:** **(If)** The stain of the organism is gram-positive $\wedge$
The morphology of the organism is coccus $\wedge$
The growth conformation of the organism is chains
**(Then)** $\Rightarrow$ The identity of the organism is streptococcus