CS 1571 Introduction to AI
Lecture 11

Knowledge Representation. Propositional logic.

Milos Hauskrecht
milos@cs.pitt.edu
5329 Sennott Square

Knowledge-based agent

Knowledge base
Inference engine

- **Knowledge base (KB):**
  - Knowledge that describe facts about the world in some formal (representational) language
  - **Domain specific**
- **Inference engine:**
  - A set of procedures that use the representational language to infer new facts from known ones or answer a variety of KB queries. Inferences typically require search.
  - **Domain independent**

Example: MYCIN

- MYCIN: an expert system for diagnosis of bacterial infections
- **Knowledge base** represents
  - Facts about a specific patient case
  - Rules describing relations between entities in the bacterial infection domain

If	1. The stain of the organism is gram-positive, and 2. The morphology of the organism is coccus, and 3. The growth conformation of the organism is chains
Then	the identity of the organism is streptococcus

- **Inference engine:**
  - manipulates the facts and known relations to answer diagnostic queries (consistent with findings and rules)

Knowledge representation

- **Objective:** express the knowledge about the world in a computer-tractable form
- **Knowledge representation languages (KRLs)**

  Key aspects:
  - **Syntax:** describes how sentences in KRL are formed in the language
  - **Semantics:** describes the meaning of sentences, what is it the sentence refers to in the real world
  - **Computational aspect:** describes how sentences and objects in KRL are manipulated in concordance with semantic conventions

Many KB systems rely on and implement some variant of logic

Logic

A formal language for expressing knowledge and for making logical inferences

Defined by:

- **A set of sentences:** A sentence is constructed from a set of primitives according to **syntactic rules**
- **A set of interpretations:** An interpretation I gives a semantic to primitives. It associates primitives with objects or values
  - I: primitives → objects/values
- **The valuation (meaning) function *V*:**
  - Assigns a value (typically the truth value) to a given sentence under some interpretation

$$V : \text{sentence} \times \text{interpretation} \rightarrow \{True, False\}$$

Propositional logic

- **The simplest logic**
- **Definition:**
  - **A proposition is a statement that is either true or false.**
- **Examples:**
  - Pitt is located in the Oakland section of Pittsburgh.
    - **(T)**

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  - 5 + 2 = 8.
    - **?**

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  - Pitt is located in the Oakland section of Pittsburgh.
    - **(T)**
  - 5 + 2 = 8.
    - **(F)**
  - It is raining today.
    - **?**

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  - **A proposition is a statement that is either true or false.**
- **Examples:**
  - Pitt is located in the Oakland section of Pittsburgh.
    - **(T)**
  - 5 + 2 = 8.
    - **(F)**
  - It is raining today.
    - **(either T or F)**

Propositional logic

- **Examples (cont.):**
  - How are you?
    - **?**

Propositional logic

- **Examples (cont.):**
  - How are you?
    - **a question is not a proposition**
  - x + 5 = 3
    - **?**

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Propositional logic

- **Examples (cont.):**
  - How are you?
    - **a question is not a proposition**
  - x + 5 = 3
    - **since x is not specified, neither true nor false**
  - 2 is a prime number.
    - **?**

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Propositional logic

- **Examples (cont.):**
  - How are you?
    - **a question is not a proposition**
  - x + 5 = 3
    - **since x is not specified, neither true nor false**
  - 2 is a prime number.
    - **(T)**

Propositional logic. Syntax

- **Formally propositional logic P:**
  - Is defined **by Syntax+interpretation+semantics of P**

Syntax:

- **Symbols (alphabet)** in P:
  - **Constants**: ***True, False***
  - **Propositional symbols**

    Examples:
    - *P*
    - *Pitt is located in the Oakland section of Pittsburgh.,*
    - *It rains outside,* etc.
  - **A set of connectives**:

$$\neg, \wedge, \vee, \Rightarrow, \Leftrightarrow$$

Propositional logic. Syntax

Sentences in the propositional logic:

- **Atomic sentences:**
  - **Constructed from constants and propositional symbols**
  - True, False are (atomic) sentences
  - P, Q or *Light in the room is on*, *It rains outside* are (atomic) sentences
- **Composite sentences:**
  - **Constructed from valid sentences via logical connectives**
  - If A, B are sentences then

    $\neg A \quad (A \wedge B) \quad (A \vee B) \quad (A \Rightarrow B) \quad (A \Leftrightarrow B)$

    or $(A \vee B) \wedge (A \vee \neg B)$

    are sentences

Propositional logic. Semantics.

The semantic gives the meaning to sentences.

the semantics in the propositional logic is defined by:

1. **Interpretation of propositional symbols and constants**
   - Semantics of atomic sentences
2. **Through the meaning of logical connectives**
   - Meaning (semantics) of composite sentences

Semantic: propositional symbols

A **propositional symbol**

- a statement about the world that is either true or false

  Examples:
  - *Pitt is located in the Oakland section of Pittsburgh*
  - *It rains outside*
  - *Light in the room is on*
- An **interpretation** maps symbols to one of the two values: ***True (T),*** or ***False (F)***, depending on whether the symbol is satisfied in the world

  I: *Light in the room is on -> **True**, It rains outside -> **False***

  I': *Light in the room is on -> **False**, It rains outside -> **False***

Semantic: propositional symbols

The **meaning (value)** of the propositional symbol for a specific interpretation is given by its interpretation

I: *Light in the room is on -> **True**, It rains outside -> **False***

V(Light in the room is on, **I***) = **True***

V(It rains outside, **I***) = **False***

I': *Light in the room is on -> **False**, It rains outside -> **False***

V(Light in the room is on, **I'***) = **False***

Interpretations		**Meanings (values)**	
Light in the room is on	*It rains outside*	*Light in the room is on*	*It rains outside*
True	True	True	True
True	False	True	False
False	True	False	True
False	False	False	False

Semantics: constants

- **The meaning (truth) of constants:**
  - True and False constants are always (under any interpretation) assigned the corresponding ***True,False*** value

$$\left.\begin{array}{l} V(True, \mathbf{I}) = \boldsymbol{True} \\ V(False, \mathbf{I}) = \boldsymbol{False} \end{array}\right\} \quad \text{For any interpretation } \mathbf{I}$$

Semantics: composite sentences.

- **The meaning (truth value) of complex propositional sentences.**
  - Determined using the standard rules of logic:

P	Q	$\neg P$	$P \wedge Q$	$P \vee Q$	$P \Rightarrow Q$	$P \Leftrightarrow Q$
True	***True***	***False***	***True***	***True***	***True***	***True***
True	***False***	***False***	***False***	***True***	***False***	***False***
False	***True***	***True***	***False***	***True***	***True***	***False***
False	***False***	***True***	***False***	***False***	***True***	***True***

Translation

Translation of English sentences to propositional logic:

(1) identify atomic sentences that are propositions

(2) Use logical connectives to translate more complex composite sentences that consist of many atomic sentences

Assume the following sentence:

- It is not sunny this afternoon and it is colder than yesterday.

Atomic sentences:

- p = It is sunny this afternoon
- q = it is colder than yesterday

Translation: $\neg p \wedge q$

Translation

Assume the following sentences:

- It is not sunny this afternoon and it is colder than yesterday.
- We will go swimming only if it is sunny.
- If we do not go swimming then we will take a canoe trip.
- If we take a canoe trip, then we will be home by sunset.

Denote:

- p = It is sunny this afternoon
- q = it is colder than yesterday
- r = We will go swimming
- s= we will take a canoe trip
- t= We will be home by sunset

Translation

Assume the following sentences:

- It is not sunny this afternoon and it is colder than yesterday. $\neg p \wedge q$
- We will go swimming only if it is sunny.
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Assume the following sentences:

- It is not sunny this afternoon and it is colder than yesterday. $\neg p \wedge q$
- We will go swimming only if it is sunny. $r \rightarrow p$
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- If we do not go swimming then we will take a canoe trip. $\neg r \rightarrow s$
- If we take a canoe trip, then we will be home by sunset.

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- If we take a canoe trip, then we will be home by sunset. $s \rightarrow t$

Denote:

- p = It is sunny this afternoon
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Contradiction and Tautology

Some composite sentences may always (under any interpretation) evaluate to a single truth value:

- **Contradiction** (always ***False***)

$$P \wedge \neg P$$

- **Tautology (**always ***True*****)**

$$P \vee \neg P$$

$$\neg(P \vee Q) \Leftrightarrow (\neg P \wedge \neg Q)$$
$$\neg(P \wedge Q) \Leftrightarrow (\neg P \vee \neg Q)$$

DeMorgan's Laws

Model, validity and satisfiability

- An interpretation **is a model for a set of sentences** if it assigns true to each sentence in the set.
- **Example 1:**
  - **Primitives:** P,Q **Sentence:** $P \vee Q$
  - **Interpretations:** **Model ?**
    - P → *True*, Q → *True* *?*

P	Q	$P \vee Q$	$(P \vee Q) \wedge \neg Q$	$((P \vee Q) \wedge \neg Q) \Rightarrow P$
True	***True***	***True***	***False***	***True***
True	***False***	***True***	***True***	***True***
False	***True***	***True***	***False***	***True***
False	***False***	***False***	***False***	***True***

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  - **Interpretations:** **Model ?**
    - P → *True*, Q → *True* *Yes*
    - P → *True*, Q → *False* *?*

P	Q	$P \vee Q$	$(P \vee Q) \wedge \neg Q$	$((P \vee Q) \wedge \neg Q) \Rightarrow P$
True	*True*	*True*	*False*	*True*
True	*False*	*True*	*True*	*True*
False	*True*	*True*	*False*	*True*
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    - P → *True*, Q → *True* *Yes*
    - P → *True*, Q → *False* *Yes*
    - P → *False*, Q → *False* *?*

P	Q	$P \vee Q$	$(P \vee Q) \wedge \neg Q$	$((P \vee Q) \wedge \neg Q) \Rightarrow P$
True	*True*	*True*	*False*	*True*
True	*False*	*True*	*True*	*True*
False	*True*	*True*	*False*	*True*
False	*False*	*False*	*False*	*True*

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- **Example 1:**
  - **Primitives:** P,Q **Sentence:** $P \vee Q$
  - **Interpretations:** **Model ?**
    - P → *True*, Q → *True* *Yes*
    - P → *True*, Q → *False* *Yes*
    - P → *False*, Q → *False* *No*

P	Q	$P \vee Q$	$(P \vee Q) \wedge \neg Q$	$((P \vee Q) \wedge \neg Q) \Rightarrow P$
True	*True*	*True*	*False*	*True*
True	*False*	*True*	*True*	*True*
False	*True*	*True*	*False*	*True*
False	*False*	*False*	*False*	*True*

Model, validity and satisfiability

- An interpretation **is a model for a set of sentences** if it assigns true to each sentence in the set.
- **Example 2:**
  - **Primitives:** P,Q
  - **Sentences:** $P \vee Q$

    $(P \vee Q) \wedge \neg Q$

    $((P \vee Q) \wedge \neg Q) \Rightarrow P$
  - **Is there a model?**

P	Q	$P \vee Q$	$(P \vee Q) \wedge \neg Q$	$((P \vee Q) \wedge \neg Q) \Rightarrow P$
True	*True*	*True*	*False*	*True*
True	*False*	*True*	*True*	*True*
False	*True*	*True*	*False*	*True*
False	*False*	*False*	*False*	*True*

Model, validity and satisfiability

- An interpretation **is a model for a set of sentences** if it assigns true to each sentence in the set.
- **Example 2:**
  - **Primitives:** P,Q
  - **Sentences:** $P \vee Q$
    $(P \vee Q) \wedge \neg Q$
    $((P \vee Q) \wedge \neg Q) \Rightarrow P$
  - **Is there a model?** **Yes** **P → *True*** **Q→ *False***

P	Q	$P \vee Q$	$(P \vee Q) \wedge \neg Q$	$((P \vee Q) \wedge \neg Q) \Rightarrow P$
True	*True*	***True***	***False***	***True***
True	*False*	***True***	***True***	***True***
False	*True*	***True***	***False***	***True***
False	*False*	***False***	***False***	***True***

Model, validity and satisfiability

- An interpretation **is a model for a set of sentences** if it assigns true to each sentence in the set.
- A sentence is **satisfiable** if it has a model;
  - There is at least one interpretation under which the sentence can evaluate to True
- **Example:**
  - **Sentence:** $(P \vee Q) \wedge \neg Q$
  - **Satisfiable?**

P	Q	$P \vee Q$	$(P \vee Q) \wedge \neg Q$	$((P \vee Q) \wedge \neg Q) \Rightarrow P$
True	*True*	*True*	***False***	*True*
True	*False*	*True*	***True***	*True*
False	*True*	*True*	***False***	*True*
False	*False*	*False*	***False***	*True*

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- **Example:**
  - **Sentence:** $(P \vee Q) \wedge \neg Q$
  - **Satisfiable? Yes** **True for** P → *True*, Q → *False*

P	Q	$P \vee Q$	$(P \vee Q) \wedge \neg Q$	$((P \vee Q) \wedge \neg Q) \Rightarrow P$
True	*True*	*True*	*False*	*True*
True	*False*	*True*	*True*	*True*
False	*True*	*True*	*False*	*True*
False	*False*	*False*	*False*	*True*

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- A sentence is **satisfiable** if it has a model;
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- A sentence is **valid** if it is ***True*** in all interpretations
  - i.e., if its negation is **not satisfiable** (leads to contradiction)

			valid sentence ?	
P	Q	$P \vee Q$	$(P \vee Q) \wedge \neg Q$	$((P \vee Q) \wedge \neg Q) \Rightarrow P$
True	*True*	*True*	*False*	*True*
True	*False*	*True*	*True*	*True*
False	*True*	*True*	*False*	*True*
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valid sentence ?

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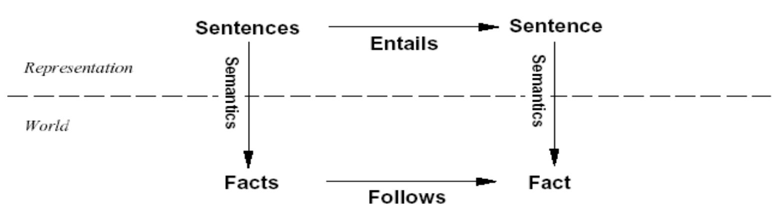
True	*True*	*True*	*False*	*True*
True	*False*	*True*	*True*	*True*
False	*True*	*True*	*False*	*True*
False	*False*	*False*	*False*	*True*

Entailment

- **Entailment** reflects the relation of one fact in the world following from the others



- Entailment $KB \models \alpha$
- Knowledge base KB entails sentence α if and only if α is true in all worlds where KB is true

Sound and complete inference

Inference is a process by which new sentences are derived from existing sentences in the KB

- the inference process is implemented on a computer

Assume an **inference procedure *i*** that

- derives a sentence α from the KB : $KB \vdash_i \alpha$

Properties of the inference procedure in terms of entailment

- **Soundness:** An inference procedure is **sound**

  If $KB \vdash_i \alpha$ then it is true that $KB \models \alpha$

- **Completeness:** An inference procedure is **complete**

  If $KB \models \alpha$ then it is true that $KB \vdash_i \alpha$

Logical inference problem

Logical inference problem:

- **Given:**
  - a knowledge base KB (a set of sentences) and
  - a sentence α (called **a theorem)**,
- **Does a KB semantically entail** α ? $KB \models \alpha$?

In other words: In all interpretations in which sentences in the KB are true, is also α true?

Question: Is there a procedure (program) that can decide this problem in a finite number of steps?

Answer: **Yes**. Logical inference problem for the propositional logic is **decidable**.

Solving logical inference problem

In the following:

How to design the procedure that answers:

$$KB \models \alpha \ ?$$

Three approaches:

- **Truth-table approach**
- **Inference rules**
- **Conversion to the inverse SAT problem**
  - **Resolution-refutation**

Truth-table approach

Problem: $KB \models \alpha \ ?$

- We need to check all possible interpretations for which the KB is true (models of KB) whether α is true for each of them

Truth table:

- enumerates truth values of sentences for all possible interpretations (assignments of True/False values to propositional symbols)

Example:

		KB		α
P	Q	$P \vee Q$	$P \Leftrightarrow Q$	$(P \vee \neg Q) \wedge Q$
True	*True*	*True*	*True*	*True*
True	*False*	*True*	*False*	*False*
False	*True*	*True*	*False*	*False*
False	*False*	*False*	*True*	*False*

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Truth table:

- enumerates truth values of sentences for all possible interpretations (assignments of True/False to propositional symbols)

Example:

		KB		α	
P	Q	$P \vee Q$	$P \Leftrightarrow Q$	$(P \vee \neg Q) \wedge Q$	
True	***True***	***True***	***True***	***True***	✓
True	***False***	***True***	***False***	***False***	
False	***True***	***True***	***False***	***False***	
False	***False***	***False***	***True***	***False***	

Truth-table approach

A two steps procedure:

1. **Generate table for all possible interpretations**
2. Check whether the sentence α evaluates to true whenever KB evaluates to true

Example: $KB = (A \vee C) \wedge (B \vee \neg C)$ $\alpha = (A \vee B)$

A	B	C	$A \vee C$	$(B \vee \neg C)$	KB	α
True	*True*	*True*				
True	*True*	*False*				
True	*False*	*True*				
True	*False*	*False*				
False	*True*	*True*				
False	*True*	*False*				
False	*False*	*True*				
False	*False*	*False*				

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Example: $KB = (A \vee C) \wedge (B \vee \neg C)$ $\alpha = (A \vee B)$

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True	*True*	*False*	*True*	*True*	*True*	*True*
True	*False*	*True*	*True*	*False*	*False*	*True*
True	*False*	*False*	*True*	*True*	*True*	*True*
False	*True*	*True*	*True*	*True*	*True*	*True*
False	*True*	*False*	*False*	*True*	*False*	*True*
False	*False*	*True*	*True*	*False*	*False*	*False*
False	*False*	*False*	*False*	*True*	*False*	*False*

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True	*True*	*True*	*True*	*True*	*True*	*True*	✓
True	*True*	*False*	*True*	*True*	*True*	*True*	✓
True	*False*	*True*	*True*	*False*	*False*	*True*	
True	*False*	*False*	*True*	*True*	*True*	*True*	✓
False	*True*	*True*	*True*	*True*	*True*	*True*	✓
False	*True*	*False*	*False*	*True*	*False*	*True*	
False	*False*	*True*	*True*	*False*	*False*	*False*	
False	*False*	*False*	*False*	*True*	*False*	*False*	

Truth-table approach

$KB = (A \vee C) \wedge (B \vee \neg C) \qquad \alpha = (A \vee B)$

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True	*True*	*False*	*True*	*True*	*True*	*True*
True	*False*	*True*	*True*	*False*	*False*	*True*
True	*False*	*False*	*True*	*True*	*True*	*True*
False	*True*	*True*	*True*	*True*	*True*	*True*
False	*True*	*False*	*False*	*True*	*False*	*True*
False	*False*	*True*	*True*	*False*	*False*	*False*
False	*False*	*False*	*False*	*True*	*False*	*False*

KB entails α

- The **truth-table approach** is **sound and complete** for the propositional logic!!

Limitations of the truth table approach.

$KB \models \alpha$?

- **What is the computational complexity of the truth table approach?**
- **?**

Limitations of the truth table approach.

$KB \models \alpha$?

- **What is the computational complexity of the truth table approach?**

Exponential in the number of the propositional symbols

2^n Rows in the table has to be filled

– the truth table is **exponential** in the number of propositional symbols (we checked all assignments)