Decision making in the presence of uncertainty

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• Many real-world problems require to choose future actions in the presence of uncertainty
• Examples: patient management, investments

Main issues:
• How to model the decision process in the computer?
• How to make decisions about actions in the presence of uncertainty?
(Stochastic) Decision tree

- Decision tree:
  - Decision node
  - Chance node
  - Outcome (value) node

Sequential (multi-step) problems

The decision tree can be build to capture multi-step decision problems:
- Choose an action
- Observe a stochastic outcome
- And repeat

How to make decisions for multi-step problems?
- Start from the leaves of the decision tree (outcome nodes)
- Compute expectations at chance nodes
- Maximize at the decision nodes

Algorithm is sometimes called expectimax
Multi-step problem example

Assume:

- Two investment periods
- Two actions: stock and bank

Selection based on expected values

- Until now: The optimal action choice was the option that maximized the expected monetary value.
- But is the expected monetary value always the quantity we want to optimize?
Selection based on expected values

- **Is the expected monetary value always the quantity we want to optimize?**
  - **Answer:** Yes, but only if we are risk-neutral.

- But what if we do not like the risk (we are risk-averse)?
  - In that case we may want to get the premium for undertaking the risk (of loosing the money)
  - **Example:** we may prefer to get $101 for sure against $102 in expectation but with the risk of loosing the money
  - **Problem:** How to model decisions and account for the risk?
  - **Solution:** use utility function, and utility theory

Utility function

- **Utility function (denoted U)**
  - Quantifies how we “value” outcomes, i.e., it reflects our preferences
  - Can be also applied to “value” outcomes other than money and gains (e.g. utility of a patient being healthy, or ill)

- **Decision making:**
  - uses expected utilities (denoted EU)

\[
EU(X) = \sum_{x \in \Omega} P(X = x)U(X = x)
\]

\[
U(X = x) \quad \text{the utility of outcome } x
\]

**Important !!!**
- Under some conditions on preferences we can always design the utility function that fits our preferences
Utility theory

- Defines axioms on preferences that involve uncertainty and ways to manipulate them.
- Uncertainty is modeled through lotteries
  - **Lottery:**
    \[ [p : A; (1 - p) : C] \]
    - Outcome A with probability p
    - Outcome C with probability \( (1-p) \)
- The following six constraints are known as the axioms of utility theory. The axioms are the most obvious semantic constraints on preferences with lotteries.

**Notation:**
- \( \succeq \) - preferable
- \( \sim \) - indifferent (equally preferable)

Axioms of the utility theory

- **Orderability:** Given any two states, the a rational agent prefers one of them, else the two as equally preferable.
  \[ (A \succ B) \lor (B \succ A) \lor (A \sim B) \]
- **Transitivity:** Given any three states, if an agent prefers A to B and prefers B to C, agent must prefer A to C.
  \[ (A \succ B) \land (B \succ C) \Rightarrow (A \succ C) \]
- **Continuity:** If some state B is between A and C in preference, then there is a \( p \) for which the rational agent will be indifferent between state B and the lottery in which A comes with probability p, C with probability \( (1-p) \).
  \[ (A \succ B \succ C) \Rightarrow \exists p \ [p : A; (1 - p) : C] \sim B \]
Axioms of the utility theory

- **Substitutability**: If an agent is indifferent between two lotteries, $A$ and $B$, then there is a more complex lottery in which $A$ can be substituted with $B$.

  \[( A \sim B ) \Rightarrow [ p : A ; (1 - p) : C ] \sim [ p : B ; (1 - p) : C ]\]

- **Monotonicity**: If an agent prefers $A$ to $B$, then the agent must prefer the lottery in which $A$ occurs with a higher probability.

  \[( A \succ B ) \Rightarrow ( p > q \iff [ p : A ; (1 - p) : B ] \succ [ q : A ; (1 - q) : B ])\]

- **Decomposability**: Compound lotteries can be reduced to simpler lotteries using the laws of probability.

  \[
  [ p : A ; (1 - p) : [ q : B ; (1 - q) : C ] ] \Rightarrow \\
  [ p : A ; (1 - p)q : B ; (1 - p)(1 - q) : C ]
  \]

Utility theory

If the agent obeys the axioms of the utility theory, then

1. There exists a real valued function $U$ such that:

   \[
   U ( A ) > U ( B ) \iff A \succ B \\
   U ( A ) = U ( B ) \iff A \sim B
   \]

2. The utility of the lottery is the expected utility, that is the sum of utilities of outcomes weighted by their probability.

   \[
   U [ p : A ; (1 - p) : B ] = pU ( A ) + (1 - p)U ( B )
   \]

3. Rational agent makes the decisions in the presence of uncertainty by maximizing its expected utility.
Utility functions

We can design a utility function that fits our preferences if they satisfy the axioms of utility theory.

- But how to design the utility function for monetary values so that they incorporate the risk?
- What is the relation between utility function and monetary values?
- Assume we loose or gain $1000.
  - Typically this difference is more significant for lower values (around $100 - 1000) than for higher values (~ $1,000,000)
- What is the relation between utilities and monetary value for a typical person?

Utility functions

- What is the relation between utilities and monetary value for a typical person?
- Concave function that flattens at higher monetary values

![Utility function graph](image-url)
Utility functions

- Expected utility of a sure outcome of 750

Assume a lottery $L$ [0.5: 500, 0.5:1000]

- Expected value of the lottery = 750
- Expected utility of the lottery $EU(L)$ is different:
  
  $EU(L) = 0.5U(500) + 0.5*U(1000)$
Utility functions

- Expected utility of the lottery $EU(\text{lottery } L) < EU(\text{sure } 750)$

Monetary value

Risk aversion – a bonus is required for undertaking the risk

Decision making with utility function

- Original problem with monetary outcomes
Decision making with the utility function

- Utility function $\log(x)$