First-order logic

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Administration announcements

Midterm:
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• In-class
• Closed book
Limitations of propositional logic

World we want to represent and reason about consists of a number of objects with variety of properties and relations among them.

**Propositional logic:**
- Represents statements about the world without reflecting this structure and without modeling these entities explicitly.

**Consequence:**
- Some knowledge is hard or impossible to encode in the propositional logic.
- Two cases that are hard to represent:
  - Statements about similar objects, relations
  - Statements referring to groups of objects.

**Example:**
Seniority of people domain

For inferences we need:

- \( \text{John is older than Mary} \land \text{Mary is older than Paul} \)
- \( \Rightarrow \text{John is older than Paul} \)
- \( \text{Jane is older than Mary} \land \text{Mary is older than Paul} \)
- \( \Rightarrow \text{Jane is older than Paul} \)

**Problem:** if we have many people and facts about their seniority, we need represent many rules like this to allow inferences.

**Possible solution:** ??

Limitations of propositional logic

• **Statements about similar objects and relations needs to be enumerated**

• **Example:** Seniority of people domain
  
  For inferences we need:
  
  \[\text{John is older than Mary} \land \text{Mary is older than Paul} \Rightarrow \text{John is older than Paul}\]
  
  \[\text{Jane is older than Mary} \land \text{Mary is older than Paul} \Rightarrow \text{Jane is older than Paul}\]

• **Problem:** if we have many people and facts about their seniority we need represent many rules like this to allow inferences

• **Possible solution:** introduce variables

  \[\text{Pers}_A \text{ is older than } \text{Pers}_B \land \text{Pers}_B \text{ is older than } \text{Pers}_C \Rightarrow \text{Pers}_A \text{ is older than } \text{Pers}_C\]


Limitations of propositional logic

• **Statements referring to groups of objects require exhaustive enumeration of objects**

• **Example:**
  
  Assume we want to express *Every student likes vacation*

  Doing this in propositional logic would require to include statements about every student

  \[\text{John likes vacation} \land\]
  
  \[\text{Mary likes vacation} \land\]
  
  \[\text{Ann likes vacation} \land\]
  
  \[\ldots\]

• **Solution:** Allow quantification in statements
First-order logic (FOL)

- More expressive than **propositional logic**

- **Eliminates deficiencies of PL by:**
  - Representing objects, their properties, relations and statements about them;
  - Introducing variables that refer to an arbitrary objects and can be substituted by a specific object
  - Introducing quantifiers allowing us to make statements over groups objects without the need to represent each of them separately

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Logic

**Logic** is defined by:

- **A set of sentences**
  - A sentence is constructed from a set of primitives according to syntax rules.

- **A set of interpretations**
  - An interpretation gives a semantic to primitives. It associates primitives with objects, values in the real world.

- **The valuation (meaning) function** $V$
  - Assigns a truth value to a given sentence under some interpretation

$$ V : \text{sentence} \times \text{interpretation} \rightarrow \{ \text{True, False} \} $$
First-order logic. Syntax.

Term – a syntactic entity for representing objects

Terms in FOL:
• **Constant symbols**: represent specific objects
  – E.g. John, France, car89
• **Variables**: represent objects of a certain type (type = domain of discourse)
  – E.g. x, y, z
• **Functions** applied to one or more terms
  – E.g. \( \text{father-of}(\text{John}) \)
  \( \text{father-of(father-of(John))} \)

First order logic. Syntax.

Sentences in FOL:
• **Atomic sentences**:
  – A **predicate symbol** applied to 0 or more terms
    Examples:
    \( \text{Red(car12)} \),
    \( \text{Sister(Amy, Jane)} \);
    \( \text{Manager(father-of(John))} \);
  
  – \( t1 = t2 \) **equivalence** of terms
    Example:
    \( \text{John} = \text{father-of(Peter)} \)
First order logic. Syntax.

Sentences in FOL:
• Complex sentences:
  • Assume $\phi, \psi$ are sentences in FOL. Then:
    - $(\phi \land \psi)$, $(\phi \lor \psi)$, $(\phi \Rightarrow \psi)$, $(\phi \Leftrightarrow \psi)$, $\neg \psi$
    and
    - $\forall x \phi$, $\exists y \phi$
  are sentences

Symbols $\exists, \forall$
- stand for the existential and the universal quantifier

Semantics. Interpretation.

An interpretation $I$ is defined by a mapping constants, predicates and function to the domain of discourse $D$ or relations on $D$
• domain of discourse: a set of objects in the world we represent and refer to;

An interpretation $I$ maps:
• Constant symbols to objects in $D$
  $I(John) = \hat{\text{John}}$
• Predicate symbols to relations, properties on $D$
  $I(\text{brother}) = \{ \langle \hat{\text{John}}, \hat{\text{Mike}} \rangle; \langle \hat{\text{Mike}}, \hat{\text{John}} \rangle; \ldots \}$
• Function symbols to functional relations on $D$
  $I(\text{father-of}) = \{ \langle \hat{\text{John}}, \hat{\text{Mike}} \rangle \rightarrow \hat{\text{Michael}}; \langle \hat{\text{Mike}}, \hat{\text{John}} \rangle \rightarrow \hat{\text{Michael}}; \ldots \}$
Semantics of sentences.

Meaning (evaluation) function:

\[ V : \text{sentence} \times \text{interpretation} \rightarrow \{ \text{True}, \text{False} \} \]

A predicate \( \text{predicate}(\text{term-1}, \text{term-2}, \text{term-3}, \text{term-n}) \) is true for the interpretation \( I \), iff the objects referred to by \( \text{term-1}, \text{term-2}, \text{term-3}, \text{term-n} \) are in the relation referred to by \( \text{predicate} \).

\[ I(\text{John}) = \begin{cases} \text{John} \\ \text{Paul} \end{cases} \]

\[ I(\text{brother}) = \{ \langle \text{John}, \text{Paul} \rangle; \langle \text{John}, \text{John} \rangle; \cdots \} \]

\[ \text{brother}(\text{John}, \text{Paul}) = \langle \text{John}, \text{Paul} \rangle \in I(\text{brother}) \]

\[ V(\text{brother}(\text{John}, \text{Paul}), I) = \text{True} \]

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Semantics of sentences.

- **Equality**
  \[ V(\text{term-1} = \text{term-2}, I) = \text{True} \]
  Iff \( I(\text{term-1}) = I(\text{term-2}) \)

- **Boolean expressions**: standard
  E.g. \[ V(\text{sentence-1} \lor \text{sentence-2}, I) = \text{True} \]
  Iff \( V(\text{sentence-1}, I) = \text{True} \) or \( V(\text{sentence-2}, I) = \text{True} \)

- **Quantifications**
  \[ V(\forall x \phi, I) = \text{True} \]
  substitution of \( x \) with \( d \)
  Iff for all \( d \in D \) \( V(\phi, I[x/d]) = \text{True} \)

  \[ V(\exists x \phi, I) = \text{True} \]
  Iff there is a \( d \in D \), s.t. \( V(\phi, I[x/d]) = \text{True} \)
Sentences with quantifiers

• **Universal quantification**

  *All Upitt students are smart*

• Assume the universe of discourse of x are Upitt students

\[ \forall x \text{ smart}(x) \]
Sentences with quantifiers

- **Universal quantification**

  *All Upitt students are smart*

  \[ \forall x \text{ smart}(x) \]

- Assume the universe of discourse of x are Upitt students
  \[ \forall x \text{ smart}(x) \]

- Assume the universe of discourse of x are students
  \[ \forall x \text{ at}(x, \text{Upitt}) \Rightarrow \text{smart}(x) \]
Sentences with quantifiers

• **Universal quantification**
  
  *All Upitt students are smart*

  $$\forall x \text{ smart}(x)$$

• Assume the universe of discourse of x are Upitt students
  $$\forall x \text{ smart}(x)$$

• Assume the universe of discourse of x are students
  $$\forall x \text{ at}(x, \text{ Upitt }) \Rightarrow \text{ smart}(x)$$

• Assume the universe of discourse of x are people
  $$\forall x \text{ student}(x) \land \text{ at}(x, \text{ Upitt }) \Rightarrow \text{ smart}(x)$$
Sentences with quantifiers

- **Universal quantification**

  \[ \forall x \ smart(x) \]
  
  *All Upitt students are smart*

- Assume the universe of discourse of \( x \) are Upitt students

- Assume the universe of discourse of \( x \) are students

- Assume the universe of discourse of \( x \) are people

  \[ \forall x \ student(x) \wedge at(x,Upitt) \Rightarrow smart(x) \]

  *Typically the universal quantifier connects with an implication*

- **Existential quantification**

  \[ \exists x \ smart(x) \]
  
  *Someone at CMU is smart*

- Assume the universe of discourse of \( x \) are CMU affiliates
Sentences with quantifiers

- **Existential quantification**

  Someone at CMU is smart

  - Assume the universe of discourse of x are CMU affiliates
    \[ \exists x \text{ smart}(x) \]

  - Assume the universe of discourse of x are people
Sentences with quantifiers

• Existential quantification

*Someone at CMU is smart*

• Assume the universe of discourse of x are CMU affiliates

\[ \exists x \ smart(x) \]

• Assume the universe of discourse of x are people

\[ \exists x \ at(x,CMU) \land smart(x) \]

Typically the existential quantifier connects with a conjunction
Translation with quantifiers

• Assume two predicates S(x) and P(x)

Universal statements typically tie with implications
• All S(x) is P(x)
  – ∀x ( S(x) → P(x) )
• No S(x) is P(x)
  – ∀x( S(x) → ¬P(x) )

Existential statements typically tie with conjunction
• Some S(x) is P(x)
  – ∃x ( S(x) ∧ P(x) )
• Some S(x) is not P(x)
  – ∃x ( S(x) ∧ ¬P(x) )

Nested quantifiers

• More than one quantifier may be necessary to capture the meaning of a statement in the predicate logic.

Example:
• There is a person who loves everybody.
• Translation:
  – Assume:
    • Variables x and y denote people
    • A predicate L(x,y) denotes: “x loves y”
• Then we can write in the predicate logic:
  ?
Nested quantifiers

- More than one quantifier may be necessary to capture the meaning of a statement in the predicate logic.

Example:
- There is a person who loves everybody.
- Translation:
  - Assume:
    - Variables \( x \) and \( y \) denote people
    - A predicate \( L(x,y) \) denotes: “\( x \) loves \( y \)”
  - Then we can write in the predicate logic:
    \[
    \exists x \forall y \ L(x,y)
    \]

Translation exercise

Suppose:
- Variables \( x,y \) denote people
- \( L(x,y) \) denotes “\( x \) loves \( y \)”.

Translate:
- Everybody loves Raymond.
Translation exercise

Suppose:
– Variables x,y denote people
– L(x,y) denotes “x loves y”.

Translate:
• Everybody loves Raymond. \( \forall x \ L(x,\text{Raymond}) \)
• Everybody loves somebody. \(?\)
Translation exercise

Suppose:
  – Variables x, y denote people
  – L(x, y) denotes “x loves y”.

Translate:
• Everybody loves Raymond. \( \forall x \ L(x, \text{Raymond}) \)
• Everybody loves somebody. \( \forall x \exists y \ L(x, y) \)
• There is somebody whom everybody loves. \( \exists y \forall x \ L(x, y) \)
• There is somebody who Raymond doesn't love. ?

Translation exercise

Suppose:
  – Variables x, y denote people
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Translate:
• Everybody loves Raymond. \( \forall x \ L(x, \text{Raymond}) \)
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• There is somebody whom everybody loves. \( \exists y \forall x \ L(x, y) \)
• There is somebody who Raymond doesn't love. \( \exists y \neg L(\text{Raymond}, y) \)
• There is somebody whom no one loves. ?
Translation exercise

Suppose:
- Variables x, y denote people
- L(x, y) denotes “x loves y”.

Translate:
- Everybody loves Raymond. $\forall x \ L(x, \text{Raymond})$
- Everybody loves somebody. $\forall x \exists y \ L(x, y)$
- There is somebody whom everybody loves. $\exists y \forall x \ L(x, y)$
- There is somebody who Raymond doesn't love. $\exists y \neg L(\text{Raymond}, y)$
- There is somebody whom no one loves. $\exists y \ \forall x \neg L(x, y)$

Order of quantifiers

- **Order of quantifiers of the same type does not matter**
  
  For all x and y, if x is a parent of y then y is a child of x
  
  $\forall x, y \ parent \ (x, y) \Rightarrow child \ (y, x)$
  
  $\forall y, x \ parent \ (x, y) \Rightarrow child \ (y, x)$

- **Order of different quantifiers changes the meaning**
  
  $\forall x \exists y \ loves \ (x, y)$
Order of quantifiers

• Order of quantifiers of the same type does not matter
  
  For all $x$ and $y$, if $x$ is a parent of $y$ then $y$ is a child of $x$
  
  $\forall x, y \text{ parent } (x, y) \Rightarrow \text{ child } (y, x)$
  
  $\forall y, x \text{ parent } (x, y) \Rightarrow \text{ child } (y, x)$

• Order of different quantifiers changes the meaning

  $\forall x \exists y \text{ loves } (x, y)$

  Everybody loves somebody

  $\exists y \forall x \text{ loves } (x, y)$

  There is someone who is loved by everyone
Connections between quantifiers

Everyone likes ice cream

∀ x likes ( x, IceCream )
Connections between quantifiers

Everyone likes ice cream

\( \forall x \text{ likes} (x, \text{IceCream} ) \)

Is it possible to convey the same meaning using an existential quantifier ?

There is no one who does not like ice cream

\( \exists x \lnot \text{ likes} (x, \text{IceCream} ) \)

A universal quantifier in the sentence can be expressed using an existential quantifier !!!
Connections between quantifiers

Someone likes ice cream

Is it possible to convey the same meaning using a universal quantifier?
Connections between quantifiers

*Someone likes ice cream*

\[ \exists x \text{ likes } (x, \text{IceCream}) \]

Is it possible to convey the same meaning using a universal quantifier?

*Not everyone does not like ice cream*

\[ \neg \forall x \neg \text{likes } (x, \text{IceCream}) \]

An existential quantifier in the sentence can be expressed using a universal quantifier!!!

Representing knowledge in FOL

Example:

**Kinship domain**

- **Objects:** people
  
  *John*, *Mary*, *Jane*, …

- **Properties:** gender
  
  *Male* (x), *Female* (x)

- **Relations:** parenthood, brotherhood, marriage
  
  *Parent* (x, y), *Brother* (x, y), *Spouse* (x, y)

- **Functions:** mother-of (one for each person x)
  
  *MotherOf* (x)
Kinship domain in FOL

**Relations between predicates and functions:** write down what we know about them; how relate to each other.

- Male and female are disjoint categories
  \[ \forall x \text{ Male} (x) \iff \neg \text{Female} (x) \]

- Parent and child relations are inverse
  \[ \forall x, y \text{ Parent} (x, y) \iff \text{Child} (y, x) \]

- A grandparent is a parent of parent
  \[ \forall g, c \text{ Grandparent}(g, c) \iff \exists p \text{ Parent}(g, p) \land \text{Parent}(p, c) \]

- A sibling is another child of one’s parents
  \[ \forall x, y \text{ Sibling} (x, y) \iff (x \neq y) \land \exists p \text{ Parent} (p, x) \land \text{Parent} (p, y) \]

- And so on ….