**Section 2.3**

**Functions**

**Definition:** Let A and B be sets. A function (mapping, map) $f$ from A to B, denoted $f: A \rightarrow B$, is a subset of $A \times B$ such that

$$\forall x [x \in A \rightarrow \exists y [y \in B \wedge <x, y> \in f]]$$

and

$$[<x, y_1> \in f \wedge <x, y_2> \in f] \rightarrow y_1 = y_2$$

Note: $f$ associates with each $x$ in A one and only one $y$ in B.

A is called the *domain* and B is called the *codomain*.

If $f(x) = y$

- $y$ is called the *image* of $x$ under $f$
- $x$ is called a *preimage* of $y$

(note there may be more than one preimage of $y$ but there is only one image of $x$).

The *range* of $f$ is the set of all images of points in A under $f$. We denote it by $f(A)$. 
If $S$ is a subset of $A$ then 

$$f(S) = \{ f(s) \mid s \text{ in } S \}.$$  

Example:

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>X</td>
</tr>
<tr>
<td>b</td>
<td>Y</td>
</tr>
<tr>
<td>c</td>
<td>Y</td>
</tr>
<tr>
<td>d</td>
<td>Z</td>
</tr>
</tbody>
</table>

- $f(a) = Z$
- the image of $d$ is $Z$
- the domain of $f$ is $A = \{a, b, c, d\}$
- the codomain is $B = \{X, Y, Z\}$
- $f(A) = \{Y, Z\}$
- the preimage of $Y$ is $b$
- the preimages of $Z$ are $a$, $c$ and $d$
- $f(\{c,d\}) = \{Z\}$
Injections, Surjections and Bijections

Let $f$ be a function from $A$ to $B$.

**Definition:** $f$ is *one-to-one* (denoted 1-1) or *injective* if preimages are unique.

Note: this means that if $a \neq b$ then $f(a) \neq f(b)$.

**Definition:** $f$ is *onto* or *surjective* if every $y$ in $B$ has a preimage.

Note: this means that for every $y$ in $B$ there must be an $x$ in $A$ such that $f(x) = y$.

**Definition:** $f$ is *bijective* if it is surjective and injective (one-to-one and onto).

Examples:

The previous Example function is neither an injection nor a surjection. Hence it is not a bijection.

\[
\begin{array}{ccc}
A & 
\rightarrow & B \\
\text{a} & \rightarrow & \text{X} \\
\text{b} & \rightarrow & \text{Y} \\
\text{c} & \rightarrow & \text{Y} \\
\text{d} & \rightarrow & \text{Z} \\
\end{array}
\]

Surjection but not an injection
Injection but not a surjection

Surjection and an injection, hence a bijection

Note: Whenever there is a bijection from A to B, the two sets must have the same number of elements or the same cardinality.

That will become our definition, especially for infinite sets.
Examples:

Let $A = B = \mathbb{R}$, the reals. Determine which are injections, surjections, bijections:

- $f(x) = x$,
- $f(x) = x^2$,
- $f(x) = x^3$,
- $f(x) = x + \sin(x)$,
- $f(x) = |x|$

Let $E$ be the set of even integers $\{0, 2, 4, 6, \ldots\}$.

Then there is a bijection $f$ from $N$ to $E$, the even nonnegative integers, defined by

$$f(x) = 2x.$$ 

Hence, the set of even integers has the same cardinality as the set of natural numbers.

**OH, NO! IT CAN’T BE....E IS ONLY HALF AS BIG!!!**

Sorry! It gets worse before it gets better.
Inverse Functions

**Definition:** Let $f$ be a bijection from $A$ to $B$. Then the *inverse* of $f$, denoted $f^{-1}$, is the function from $B$ to $A$ defined as

$$f^{-1}(y) = x \iff f(x) = y$$

Example:

Let $f$ be defined by the diagram:

![Diagram](image)

Note: No inverse exists unless $f$ is a bijection.
**Definition:** Let $S$ be a subset of $B$. Then

$$f^{-1}(S) = \{ x \mid f(x) \in S \}$$

Note: $f$ need not be a bijection for this definition to hold.

**Example:**

Let $f$ be the following function:

$$f^{-1}(\{Z\}) = \{c, d\}$$

$$f^{-1}(\{X, Y\}) = \{a, b\}$$
**Composition**

**Definition:** Let \( f: B \rightarrow C, \) \( g: A \rightarrow B. \) The *composition of \( f \) with \( g, * \) denoted \( f \circ g, \) is the function from \( A \) to \( C \) defined by

\[
f \circ g(x) = f(g(x))
\]

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Examples:

\[
\begin{array}{cccccc}
A & g & B & f & C \\
| a | o | v | o | h \\
| b | o | w | o | i \\
| c | o | x | o | j \\
| d | o | y | o |
\end{array}
\]

\[
\begin{array}{cccccc}
A & f \circ g & C \\
| a | o | h \\
| b | o | i \\
| c | o | j \\
| d |
\end{array}
\]

If \( f(x) = x^2 \) and \( g(x) = 2x + 1, \) then \( f(g(x)) = (2x+1)^2 \) and \( g(f(x)) = 2x^2 + 1 \)
**Definition:** The

floor function,

\[ f(x) = \lfloor x \rfloor \text{ or } f(x) = \text{floor}(x), \]

is the largest integer less than or equal to \( x \).

The

ceiling function,

\[ f(x) = \lceil x \rceil \text{ or } f(x) = \text{ceiling}(x), \]

is the smallest integer greater than or equal to \( x \).

Examples: \( \lfloor 3.5 \rfloor = 3, \lceil 3.5 \rceil = 4. \)

Note: the floor function is equivalent to truncation for positive numbers.

Example:

Suppose \( f: B \rightarrow C, \ g: A \rightarrow B \) and \( f \circ g \) is injective.

What can we say about \( f \) and \( g \)?

- We know that if \( a \neq b \) then \( f(g(a)) \neq f(g(b)) \) since the composition is injective.
- Since $f$ is a function, it cannot be the case that $g(a) = g(b)$ since then $f$ would have two different images for the same point.

- Hence, $g(a) \neq g(b)$

It follows that $g$ must be an injection.

However, $f$ need not be an injection (you show).