Section 2.2
Set Operations

Propositional calculus and set theory are both instances of an algebraic system called a

Boolean Algebra.

The operators in set theory are defined in terms of the corresponding operator in propositional calculus

As always there must be a universe \( U \). All sets are assumed to be subsets of \( U \)

**Definition:** Two sets \( A \) and \( B \) are equal, denoted \( A = B \), iff

\[
\forall x [x \in A \iff x \in B].
\]

Note: By a previous logical equivalence we have

\[
A = B \iff \forall x [(x \in A \rightarrow x \in B) \land (x \in B \rightarrow x \in A)]
\]

or

\[
A = B \iff A \subseteq B \text{ and } B \subseteq A
\]
Definitions:

• The union of A and B, denoted $A \cup B$, is the set
  $$\{x \mid x \in A \lor x \in B\}$$

• The intersection of A and B, denoted $A \cap B$, is the set
  $$\{x \mid x \in A \land x \in B\}$$

Note: If the intersection is void, A and B are said to be disjoint.

• The complement of A, denoted $\overline{A}$, is the set
  $$\{x \mid \neg(x \in A)\}$$

Note: Alternative notation is $A^c$, and $\{x \mid x \notin A\}$.

• The difference of A and B, or the complement of B relative to A, denoted $A - B$, is the set
  $$A \cap \overline{B}$$

Note: The (absolute) complement of A is $U - A$.

• The symmetric difference of A and B, denoted $A \oplus B$, is the set
  $$(A - B) \cup (B - A)$$
Examples: $U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

$A = \{1, 2, 3, 4, 5\}$, $B = \{4, 5, 6, 7, 8\}$. Then

- $A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8\}$
- $A \cap B = \{4, 5\}$
- $\overline{A} = \{0, 6, 7, 8, 9, 10\}$
- $\overline{B} = \{0, 1, 2, 3, 9, 10\}$
- $A - B = \{1, 2, 3\}$
- $B - A = \{6, 7, 8\}$
- $A \oplus B = \{1, 2, 3, 6, 7, 8\}$
Venn Diagrams

A useful geometric visualization tool (for 3 or less sets)

- The Universe U is the rectangular box
- Each set is represented by a circle and its interior
- All possible combinations of the sets must be represented

For 2 sets

For 3 sets

Shade the appropriate region to represent the given set operation.

Set Identities

Set identities correspond to the logical equivalences.
Example:

The complement of the union is the intersection of the complements:

$$A \cup B = \overline{A} \cap \overline{B}$$

Proof: To show:

$$\forall x [x \in A \cup B \iff x \in \overline{A} \cap \overline{B}]$$

To show two sets are equal we show for all x that x is a member of one set if and only if it is a member of the other.

We now apply an important rule of inference (defined later) called

**Universal Instantiation**

In a proof we can eliminate the universal quantifier which binds a variable if we do not assume anything about the variable other than it is an arbitrary member of the Universe. We can then treat the resulting predicate as a proposition.

We say

'Let x be arbitrary.'

Then we can treat the predicates as propositions:
<table>
<thead>
<tr>
<th>Assertion</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x \in \overline{A \cup B} \iff x \notin [A \cup B]$</td>
<td>Def. of complement</td>
</tr>
<tr>
<td>$x \notin A \cup B \iff \neg [x \in A \cup B]$</td>
<td>Def. of $\notin$</td>
</tr>
<tr>
<td>$\iff \neg [x \in A \lor x \in B]$</td>
<td>Def. of union</td>
</tr>
<tr>
<td>$\iff \neg x \in A \land \neg x \in B$</td>
<td>DeMorgan's Laws</td>
</tr>
<tr>
<td>$\iff x \notin A \land x \notin B$</td>
<td>Def. of $\notin$</td>
</tr>
<tr>
<td>$\iff x \in \overline{A} \land x \in \overline{B}$</td>
<td>Def. of complement</td>
</tr>
<tr>
<td>$\iff x \in \overline{A \cap B}$</td>
<td>Def. of intersection</td>
</tr>
</tbody>
</table>

Hence

$$x \in \overline{A \cup B} \iff x \in \overline{A} \cap \overline{B}$$

is a tautology.

Since

- x was arbitrary

- we have used only logically equivalent assertions and definitions

we can apply another rule of inference called
**Universal Generalization**

We can apply a universal quantifier to bind a variable if we have shown the predicate to be true for all values of the variable in the Universe.

and claim the assertion is true for all x, i.e.,

\[ \forall x [x \in A \cup B \iff x \in A \cap B] \]

Q. E. D. (an abbreviation for the Latin phrase “Quod Erat Demonstrandum” - “which was to be demonstrated” used to signal the end of a proof)

Note: As an alternative which might be easier in some cases, use the identity

\[ A = B \iff [A \subseteq B \text{ and } B \subseteq A] \]

Example:

Show \( A \cap (B - A) = \emptyset \)

The void set is a subset of every set. Hence,

\[ A \cap (B - A) \supseteq \emptyset \]

Therefore, it suffices to show
\[ A \cap (B - A) \subseteq \varnothing \]

or

\[ \forall x [x \in A \cap (B - A) \rightarrow x \in \varnothing] \]

So as before we say 'let x be arbitrary'.

Show

\[ x \in A \cap (B - A) \rightarrow x \in \varnothing \]

is a tautology.

But the consequent is always false.

Therefore, the antecedent better always be false also.

Apply the definitions:

<table>
<thead>
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</tr>
</thead>
<tbody>
<tr>
<td>( x \in A \cap (B - A) \leftrightarrow x \in A \land x \in (B - A) )</td>
<td>Def. of ( \cap )</td>
</tr>
<tr>
<td>( \leftrightarrow x \in A \land (x \in B \land x \notin A) )</td>
<td>Def. of (-)</td>
</tr>
<tr>
<td>( \leftrightarrow (x \in A \land x \notin A) \land x \in B )</td>
<td>Props of 'and'</td>
</tr>
<tr>
<td>( \leftrightarrow 0 \land x \in B )</td>
<td>Table 6</td>
</tr>
<tr>
<td>( \leftrightarrow 0 )</td>
<td>Domination</td>
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</tbody>
</table>

Hence, because \( P \land \neg P \) is always false, the implication is a tautology.

The result follows by Universal Generalization.

Q. E. D.
Union and Intersection of Indexed Collections

Let $A_1, A_2, \ldots, A_n$ be an indexed collection of sets.

Union and intersection are associative (because 'and' and 'or' are) we have:

\[
\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots \cup A_n
\]

and

\[
\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \ldots \cap A_n
\]

Examples:

Let

\[ A_i = [i, \infty), 1 \leq i < \infty \]

\[
\bigcup_{i=1}^{n} A_i = [1, \infty)
\]

\[
\bigcap_{i=1}^{n} A_i = [n, \infty)
\]