Binary relations establish a relationship between elements of two sets

**Definition:** Let A and B be two sets. A binary relation from A to B is a subset of $A \times B$.

In other words, a binary relation $R$ is a set of ordered pairs $(a_i, b_i)$ where $a_i \in A$ and $b_i \in B$.

**Notation:** We say that
- $a R b$ if $(a, b) \in R$
- $a \not R b$ if $(a, b) \notin R$

**Definition:** A relation on the set $A$ is a relation from $A$ to $A$. That is, a relation on the set $A$ is a subset of $A \times A$.

What is an equivalence relation?

**Informally:** An equivalence relation partitions elements of a set into classes of “equivalent” objects.

**Formally:** A relation on a set $A$ is called an equivalence relation if it is reflexive, symmetric, and transitive.

- How can a relation define equivalent objects if an element isn’t equivalent to itself?
- If $x$ is equivalent to $y$, and $y$ is equivalent to $z$, shouldn’t $x$ also be equivalent to $z$?

**Definition:** Two elements $a$ and $b$ that are related by some equivalence relation are called equivalent. We denote this by $a \sim b$ (or $a \sim_R b$).
**Example: Comparing Magnitudes**

*Example:* Let $R$ be the relation on the set of integers such that $a R b$ if and only if $a = b$ or $a = -b$. Is $R$ an equivalence relation?

Intuition says yes, so let’s verify:

- Is $R$ reflexive?
- Is $R$ symmetric?
- Is $R$ transitive?

**Conclusion:** Since $R$ is symmetric, reflexive, and transitive, we know that $R$ is an equivalence relation.

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**Congruence Modulo $m$**

*Example:* Let $m$ be a positive integer greater than 1. Show that $R = \{(a,b) \mid a \equiv b \pmod{m}\}$ is an equivalence relation.

**Solution:**

- **Recall:** $a \equiv b \pmod{m} \leftrightarrow m \mid (a - b)$
- Is $R$ reflexive?
  - $a \equiv a \pmod{m} \leftrightarrow m \mid (a - a)$
  - $m \mid 0$ since $0 = 0 \times m$
  - Yes, $R$ is reflexive
- Is $R$ symmetric?
  - If $a \equiv b \pmod{m}$, then $m \mid (a - b)$, so $(a - b) = km$ for some $k$
  - Note that $(b - a) = -km$
  - So $b \equiv a \pmod{m}$ and $R$ is symmetric
- Is $R$ transitive?
  - $a \equiv b \pmod{m}$ means that $(a - b) = km$, so $a = km + b$
  - $b \equiv c \pmod{m}$ means that $(b - c) = jm$, so $c = b - jm$
  - Note that $a - c = (km + b) - (b - jm) = km + jm = (k+j)m$
  - Since $m \mid (a - c)$, $a \equiv c \pmod{m}$, and $R$ is transitive
- **Conclusion:** $R$ is an equivalence relation
What about the “divides” relation?

**Example:** Is the “divides” relation on positive integers an equivalence relation?

**Solution:**
- Reflexive?
- Symmetric?
- Transitive?

**Conclusion:** Since the “divides” relation is not symmetric, it cannot be an equivalence relation.

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String Length

**Example:** Suppose that $R$ is the relation on the set of strings of English letters such that $a R b$ if and only if $l(a) = l(b)$, where $l(x)$ is the length of string $x$. Is $R$ an equivalence relation?

**Solution:**
- Reflexive?
- Symmetric?
- Transitive?

$R c.$

**Conclusion:** $R$ is an equivalence relation.
Magnitude of differences

**Example:** Let R be the relation on the set of real numbers such that \( x \, R \, y \) iff \( x \) and \( y \) are real numbers that differ by less than 1, i.e., \(|x - y| < 1\). Is \( R \) an equivalence relation?

**Solution:**
- First, a few test cases:
  - \( 1.1 \, R \, 2.0 \)?: Yes, since
  - \( 1.1 \, R \, 3.0 \)?: No, since 
  - \( 2.0 \, R \, 2.5 \)?: Yes, since 
- Reflexive?
- Symmetric?
- Transitive?
- Conclusion: Since \( R \) is not transitive, it cannot be an equivalence relation.

What is an equivalence class?

**Definition:** Let \( R \) be an equivalence relation on a set \( A \). The set of all elements that are related to some element \( a \) is called the equivalence class of \( a \).

**Note:** We denote the equivalence class of element \( a \) under relation \( R \) as \([a]_R\). If only one relation is being considered, we can drop the subscript and denote the equivalence class of \( a \) as \([a]\).

**Example:** What are the equivalence classes of 0 and 1 under congruence modulo 4?
- \([0]\) contains all integers \( x \) such that \( x \equiv 0 \pmod{4} \)
- \([1]\) contains all integers \( x \) such that \( x \equiv 1 \pmod{4} \)
- So \([0]\) = \([-8, -4, 0, 4, 8, ...]\)
- And \([1]\) = \([-7, -3, 1, 5, 9, ...]\)
Variable names in C

Example: Some compilers for the C programming language truncate variable names after the first 31 characters. As a result, any two variable names that agree in the first 31 characters are considered to be identical. What are the equivalence classes of the variable names “Number_of_tropical_storms”, “Number_of_named_tropical_storms”, and “Number_of_named_tropical_storms_in_the_Atlantic_in_2005”?

Solution:
- [Number_of_tropical_storms] =
- [Number_of_named_tropical_storms] =
- [Number_of_named_tropical_storms_in_the_Atlantic_in_2005] =

An equivalence relation divides a set into disjoint subsets

(Contrived) Example: At State University, a student can either major in computer science or art history, but not both. Let R be the relation defined such that a R b if a and b are in the same major.

Observations:
- R is an equivalence relation (Why?)
- R breaks the set S of all students into two subsets:
  - C = Students majoring in computer science
  - A = Students majoring in art history
- No student in C is also in A
- No student in A is also in C
- C and A are equivalence classes of S
Equivalence classes are either equal or disjoint

**Theorem:** If R is an equivalence relation on some set A, then the following three statements are equivalent: (i) a R b, (ii) \([a] = [b]\), and (iii) \([a] \cap [b] \neq \emptyset\).

**Proof:**

- To prove this, we'll prove that (i) \(\rightarrow\) (ii), (ii) \(\rightarrow\) (iii), and (iii) \(\rightarrow\) (i)

  - (i) \(\rightarrow\) (ii)
    - Assume that a R b
    - To prove that \([a] = [b]\), we will show that \([a] \subseteq [b]\) and \([b] \subseteq [a]\)
    - Suppose that \(c \in [a]\), then a R c
    - Since a R b and R is symmetric, we have that b R a
    - Since R is transitive, we have that b R a and a R c, so b R c
    - This means that \(c \in [b]\) and thus that \([a] \subseteq [b]\)
    - The proof that \([b] \subseteq [a]\) is identical

**Proof (cont.):**

- (ii) \(\rightarrow\) (iii)
  - Assume that \([a] = [b]\)
  - \([a] \cap [b]\) is non-empty since \(a \in [a]\)

- (iii) \(\rightarrow\) (i)
  - Assume that \([a] \cap [b] \neq \emptyset\)
  - This means that there exists some element \(c \in [a] \cap [b]\)
  - So, a R c and b R c
  - By symmetry, we have that c R b
  - By transitivity, we have that a R c and c R b means a R b

Since (i) \(\rightarrow\) (ii), (ii) \(\rightarrow\) (iii), and (iii) \(\rightarrow\) (i), all three statements are equivalent.
**Equivalence classes partition a set**

*Definition:* A partition of a set $S$ is a collection of disjoint subsets that have $S$ as their union.

![Diagram showing partition of a set](image)

*Observation:* The equivalence classes of a set partition that set.
- $U_{eq} [a] = A$ since each $a \in A$ is in its own equivalence class
- By our theorem, we know that either $[a] = [b]$, or $[a] \cap [b] = \emptyset$

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**The integers (mod m), redux**

*Example:* What are the sets in the partition produced by the equivalence relation equivalence mod 4?

*Solution:*
- $[0] = \{\ldots, -8, -4, 0, 4, 8, \ldots\}$
- $[1] = \{\ldots, -7, -3, 1, 5, 9, \ldots\}$
- $[2] = \{\ldots, -6, -2, 2, 6, 10, \ldots\}$
- $[3] = \{\ldots, -5, -1, 3, 7, 11, \ldots\}$

- Note that each integer is in one of these sets, and each set is disjoint. Thus, these equivalence classes partition the set $\mathbb{Z}$. 
Conversely, a partition of a set describes an equivalence relation

**Example:** List the ordered pairs in the equivalence relation R produced by the partition \( A = \{1, 2, 3\} \), \( B = \{4, 5\} \), \( C = \{6\} \) of \( S = \{1, 2, 3, 4, 5, 6\} \).

**Solution:**
- From \( A = \{1, 2, 3\} \) we have
  - \((1,1), (1,2), (1,3) \in R\)
  - \((2,1), (2,2), (2,3) \in R\)
  - \((3,1), (3,2), (3,3) \in R\)
- From \( B = \{4, 5\} \) we have
  - \((4,4), (4,5) \in R\)
  - \((5,4), (5,5) \in R\)
- From \( C = \{6\} \) we have
  - \((6,6) \in R\)

\[ \text{Group Work!} \]

**Problem 1:** Which of the following relations on \( \{0, 1, 2, 3\} \) are equivalence relations? Which properties are lacking from those relations that are not equivalence relations?
1. \[\{(0, 0), (1,1), (2,2), (3,3)\}\]
2. \[\{(0,0), (0,2), (2,0), (2,2), (2,3), (3,2), (3,3)\}\]

**Problem 2:** Which of these collections of sets are partitions of the set \( S = \{1, 2, 3, 4, 5, 6\} \)?
1. \[\{1,2\}, \{2,3,4\}, \{4,5,6\}\]
2. \[\{2,4,6\}, \{1,3,5\}\]
3. \[\{1,4,5\}, \{2, 6\}\]