Today

Relations
- Binary relations and properties
- Relationship to functions

n-ary relations
- Definitions

Binary relations establish a relationship between elements of two sets

Definition: Let A and B be two sets. A binary relation from A to B is a subset of $A \times B$.

In other words, a binary relation $R$ is a set of ordered pairs $(a_i, b_i)$ where $a_i \in A$ and $b_i \in B$.

Notation: We say that
- $a R b$ if $(a, b) \in R$
- $a \not R b$ if $(a, b) \not \in R$
Example: Course Enrollments

Let’s say that Alice and Bob are taking CS 441. Alice is also taking Math 336. Furthermore, Charlie is taking Art 212 and Business 444. Define a relation R that represents the relationship between people and classes.

Solution:
- Let the set P denote people, so P = {Alice, Bob, Charlie}
- Let the set C denote classes, so C = {CS 441, Math 336, Art 212, Business 444}
- By definition R ⊆ P × C
- From the above statement, we know that
  - (Alice, CS 441) ∈ R
  - (Bob, CS 441) ∈ R
  - (Alice, Math 336) ∈ R
  - (Charlie, Art 212) ∈ R
  - (Charlie, Business 444) ∈ R
- So, R = {(Alice, CS 441), (Bob, CS 441), (Alice, Math 336), (Charlie, Art 212), (Charlie, Business 444)}

A relation can also be represented as a graph

Let’s say that Alice and Bob are taking CS 441. Alice is also taking Math 336. Furthermore, Charlie is taking Art 212 and Business 444. Define a relation R that represents the relationship between people and classes.
A relation can also be represented as a table

Let’s say that Alice and Bob are taking CS 441. Alice is also taking Math 336. Furthermore, Charlie is taking Art 212 and Business 444. Define a relation R that represents the relationship between people and classes.

<table>
<thead>
<tr>
<th></th>
<th>Art 212</th>
<th>Business 444</th>
<th>CS 441</th>
<th>Math 336</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alice</td>
<td></td>
<td></td>
<td>X</td>
<td>X</td>
</tr>
<tr>
<td>Bob</td>
<td></td>
<td></td>
<td></td>
<td>X</td>
</tr>
<tr>
<td>Charlie</td>
<td>X</td>
<td>X</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Name of the relation

(Bob, CS 441) ∈ R

Elements of C (i.e., courses)

Elements of P (i.e., people)

Wait, doesn’t this mean that relations are the same as functions?

Not quite... Recall the following definition from Lecture #9.

**Definition:** Let A and B be nonempty sets. A function, f, is an assignment of exactly one element of set B to each element of set A.

This would mean that, e.g., a person only be enrolled in one course!

Reconciling this with our definition of a relation, we see that

1. Every function is also a relation
2. Not every relation is a function

Let’s see some quick examples...
Short and sweet...

1. Consider \( f : S \rightarrow G \)
   - Clearly a function
   - Can also be represented as the relation \( R = \{ (\text{Anna}, C), (\text{Brian}, A), (\text{Christine}, A) \} \)

We can also define binary relations on a single set

**Definition:** A relation on the set \( A \) is a relation from \( A \) to \( A \). That is, a relation on the set \( A \) is a subset of \( A \times A \).

**Example:** Let \( A \) be the set \{1, 2, 3, 4\}. Which ordered pairs are in the relation \( R = \{ (a, b) \mid a \text{ divides } b \} \)?

**Solution:**
- 1 divides everything
- 2 divides itself and 4
- 3 divides itself
- 4 divides itself

So, \( R = \{ (1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4) \} \)
**Representing the last example as a graph...**

**Example:** Let A be the set \{1, 2, 3, 4\}. Which ordered pairs are in the relation \( R = \{(a, b) \mid a \text{ divides } b\} \)?

![Graph diagram](image)

**Tell me what you know...**

**Question:** Which of the following relations contain each of the pairs \((1,1), (1,2), (2,1), (1,-1), \text{ and } (2,2)\)?

- \( R_1 = \{(a, b) \mid a \leq b\} \)
- \( R_2 = \{(a, b) \mid a > b\} \)
- \( R_3 = \{(a, b) \mid a = b \text{ or } a = -b\} \)
- \( R_4 = \{(a, b) \mid a = b\} \)
- \( R_5 = \{(a, b) \mid a = b + 1\} \)
- \( R_6 = \{(a, b) \mid a + b \leq 3\} \)

**Answer:**

<table>
<thead>
<tr>
<th></th>
<th>((1,1))</th>
<th>((1,2))</th>
<th>((2,1))</th>
<th>((1,-1))</th>
<th>((2,2))</th>
</tr>
</thead>
<tbody>
<tr>
<td>( R_1 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( R_2 )</td>
<td></td>
<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>( R_3 )</td>
<td></td>
<td></td>
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<tr>
<td>( R_4 )</td>
<td></td>
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<tr>
<td>( R_5 )</td>
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<td></td>
</tr>
<tr>
<td>( R_6 )</td>
<td></td>
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</tbody>
</table>

These are all relations on an infinite set!
Properties of Relations

**Definition:** A relation R on a set A is reflexive if \((a,a) \in R\) for every \(a \in A\).

**Note:** Our “divides” relation on the set \(A = \{1,2,3,4\}\) is reflexive.

Every \(a \in A\) divides itself!

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Properties of Relations

**Definition:** A relation R on a set A is symmetric if \((b,a) \in R\) whenever \((a,b) \in R\) for every \(a,b \in A\). If R is a relation in which \((a,b) \in R\) and \((b,a) \in R\) implies that \(a=b\), we say that R is antisymmetric.

**Mathematically:**
- Symmetric: \(\forall a \forall b ((a,b) \in R \rightarrow (b,a) \in R)\)
- Antisymmetric: \(\forall a \forall b ((a,b) \in R \wedge (b,a) \in R) \rightarrow (a = b))\)

**Examples:**
- Symmetric: \(R = \{(1,1), (1,2), (2,1), (2,3), (3,2), (1,4), (4,1), (4,4)\}\)
- Antisymmetric: \(R = \{(1,1), (1,2), (1,3), (1,4), (2,4), (3,3), (4,4)\}\)
### Symmetric and Antisymmetric Relations

**Definition:**

- A relation $R$ on a set $A$ is **symmetric** if for all $a, b \in A$, whenever $(a, b) \in R$, then $(b, a) \in R$.
- A relation $R$ on a set $A$ is **antisymmetric** if for all $a, b \in A$, whenever $(a, b) \in R$ and $(b, a) \in R$, then $a = b$.

**Example:**

- **R** = \{(1,1), (1,2), (2,1), (2,3), (3,2), (1,4), (4,1), (4,4)\}

  - **Symmetric relation**
    - Diagonal axis of symmetry
    - Not all elements on the axis of symmetry need to be included in the relation

- **R** = \{(1,1), (1,2), (1,3), (1,4), (2,4), (3,3), (4,4)\}

  - **Asymmetric relation**
    - No axis of symmetry
    - Only symmetry occurs on diagonal
    - Not all elements on the diagonal need to be included in the relation

### Properties of Relations

**Definition:** A relation $R$ on a set $A$ is **transitive** if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$ for every $a, b, c \in A$.

**Note:** Our “divides” relation on the set $A = \{1,2,3,4\}$ is transitive.

- $1$ divides $2$
- $1$ divides $4$
- $2$ divides $4$
- $3$ divides $4$

This isn’t terribly interesting, but it is transitive nonetheless....

More common transitive relations include equality and comparison operators like $<$, $>$, $\leq$, and $\geq$. 

Examples, redux

**Question:** Which of the following relations are reflexive, symmetric, antisymmetric, and/or transitive?

- $R_1 = \{(a,b) \mid a \leq b\}
- $R_2 = \{(a,b) \mid a > b\}
- $R_3 = \{(a,b) \mid a = b \text{ or } a = -b\}
- $R_4 = \{(a,b) \mid a = b\}
- $R_5 = \{(a,b) \mid a = b + 1\}
- $R_6 = \{(a,b) \mid a + b \leq 3\}

**Answer:**

<table>
<thead>
<tr>
<th></th>
<th>Reflexive</th>
<th>Symmetric</th>
<th>Antisymmetric</th>
<th>Transitive</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_1$</td>
<td>✓</td>
<td></td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>$R_2$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$R_3$</td>
<td>✓</td>
<td></td>
<td></td>
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<tr>
<td>$R_4$</td>
<td>✓</td>
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<tr>
<td>$R_5$</td>
<td></td>
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<tr>
<td>$R_6$</td>
<td>No</td>
<td>Yes</td>
<td>No</td>
<td>No</td>
</tr>
</tbody>
</table>

Relations can be combined using set operations

**Example:** Let $R$ be the relation that pairs students with courses that they have taken. Let $S$ be the relation that pairs students with courses that they need to graduate. What do the relations $R \cup S$, $R \cap S$, and $S - R$ represent?

**Solution:**

- $R \cup S =$ All pairs $(a,b)$ where
  - Student $a$ has taken course $b$ OR
  - Student $a$ needs to take course $b$ to graduate

- $R \cap S =$ All pairs $(a,b)$ where
  - Student $a$ has taken course $b$ AND
  - Student $a$ needs course $b$ to graduate

- $S - R =$ All pairs $(a,b)$ where
  - Student $a$ needs to take course $b$ to graduate BUT
  - Student $a$ has not yet taken course $b$
Relations can be combined using functional composition

**Definition:** Let $R$ be a relation from the set $A$ to the set $B$, and $S$ be a relation from the set $B$ to the set $C$. The **composite** of $R$ and $S$ is the relation of ordered pairs $(a, c)$, where $a \in A$ and $c \in C$ for which there exists an element $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$. We denote the composite of $R$ and $S$ by $R \circ S$.

**Example:** What is the composite relation of $R$ and $S$?

$R: \{1,2,3\} \to \{1,2,3,4\}$  \hspace{1cm} $S: \{1,2,3,4\} \to \{0,1,2\}$

$R = \{(1,1), (1,4), (2,3), (3,1), (3,4)\}$  \hspace{1cm} $S = \{(1,0), (2,0), (3,1), (3,2), (4,1)\}$

So: $R \circ S = \{(1,0), (3,0), (1,1), (3,1), (2,1), (2,2)\}$

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**Group Work!**

**Problem 1:** List the ordered pairs of the relation $R$ from $A = \{0,1,2,3,4\}$ to $B = \{0,1,2,3\}$ where $(a,b) \in R$ iff $a + b = 4$.

**Problem 2:** Is the relation $\{(2,4), (4,2)\}$ on the set $\{1,2,3,4\}$ reflexive, symmetric, antisymmetric, and/or transitive?
We can also “relate” elements of more than two sets

**Definition:** Let $A_1, A_2, \ldots, A_n$ be sets. An $n$-ary relation on these sets is a subset of $A_1 \times A_2 \times \ldots \times A_n$. The sets $A_1, A_2, \ldots, A_n$ are called the **domains** of the relation, and $n$ is its **degree**.

**Example:** Let $R$ be the relation on $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}^*$ consisting of triples $(a, b, m)$ where $a \equiv b \pmod{m}$.

- What is the degree of this relation?
- What are the domains of this relation?
- Are the following tuples in this relation?
  - $(8,2,3)$
  - $(-1,9,5)$
  - $(11,0,6)$

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**Final Thoughts**

- Relations allow us to represent and reason about the relationships between sets in a more general way than functions did.