Today's Topic: Propositional Logic

- What is a proposition?
- Logical connectives and truth tables
- Translating between English and propositional logic

Logic is the basis of all mathematical and analytical reasoning

Given a collection of known truths, logic allows us to deduce new truths

**Example**

Base facts:
- If it is raining, I will not go outside
- If I am inside, Stephanie will come over
- Stephanie and I always play Scrabble if we are together during the weekend
- Today is rainy Saturday

**Conclusion:** Stephanie and I will play Scrabble today

Logic allows us to advance mathematics through an iterative process of **conjecture** and **proof**
Propositional logic is a very simple logic

**Definition:** A proposition is a precise statement that is either true or false, but not both.

Examples:
- \(2 + 2 = 4\) (true)
- All dogs have 3 legs (false)
- \(x^2 < 0\) (false)
- Washington, D.C. is the capital of the USA (true)

Not all statements are propositions

- Marcia is pretty
  - “Pretty” is a subjective term.

- \(x^3 < 0\)
  - True if \(x < 0\), false otherwise.

- Springfield is the capital
  - True in Illinois, false in Massachusetts.
We can use logical connectives to build complex propositions

We will discuss the following logical connectives:
- ¬ (not)
- ∧ (conjunction / and)
- ∨ (disjunction / or)
- ⊕ (exclusive or)
- → (implication)
- ↔ (biconditional)

Negation

The negation of a proposition is true iff the proposition is false

The truth table for negation

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<thead>
<tr>
<th>p</th>
<th>¬p</th>
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<tbody>
<tr>
<td></td>
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</table>

One row for each possible value of “what we know”

What we know

What we want to know
Negation Examples

Negate the following propositions

- Today is Monday
- \(21 \times 2 = 42\)

What is the truth value of the following propositions

- \(\neg(9\text{ is a prime number})\)
- \(\neg(\text{Pittsburgh is in Pennsylvania})\)

Conjunction

The conjunction of two propositions is true iff both propositions are true

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<thead>
<tr>
<th>p</th>
<th>q</th>
<th>p \land q</th>
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<tbody>
<tr>
<td>T</td>
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The truth table for conjunction

\(2^2 = 4\text{ rows since we know both } p \text{ and } q!\)
Disjunction

The disjunction of two propositions is true if at least one proposition is true.

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<th>p ∨ q</th>
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<tbody>
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The truth table for disjunction

Conjunction and disjunction examples

Let:

- $p \equiv x^2 > 0$
- $q \equiv \text{A lion weighs less than a mouse}$
- $r \equiv 10 < 7$
- $s \equiv \text{Pittsburgh is located in Pennsylvania}$

What are the truth values of these expressions:

- $p \land q$
- $p \land s$
- $p \lor q$
- $q \lor r$
### Exclusive or (XOR)

The **exclusive or** of two propositions is true if *exactly one* proposition is true.

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<th>p ⊕ q</th>
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<tbody>
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The truth table for exclusive or

**Note:** Exclusive or is typically used to natural language to identify *choices*. For example “You may have a soup or salad with your entree.”

### Implication

The **implication** \( p \rightarrow q \) is **false** if \( p \) is **true** and \( q \) is **false**, and **true** otherwise.

**Terminology**

- \( p \) is called the hypothesis
- \( q \) is called the conclusion

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<th>q</th>
<th>( p \rightarrow q )</th>
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The truth table for implication
The implication $p \rightarrow q$ can be read in a number of (equivalent) ways:

- If $p$ then $q$
- $p$ only if $q$
- $p$ is sufficient for $q$
- $q$ whenever $p$

**Implication examples**

Let:

- $p \equiv$ Jane gets a 100% on her final
- $q \equiv$ Jane gets an A

What are the truth values of these implications:

- $p \rightarrow q$
- $q \rightarrow p$
Other conditional statements

Given an implication $p \rightarrow q$:
- $q \rightarrow p$ is its converse
- $\neg q \rightarrow \neg p$ is its contrapositive
- $\neg p \rightarrow \neg q$ is its inverse

Note: An implication and its contrapositive always have the same truth value

Biconditional

The biconditional $p \leftrightarrow q$ is true if and only if $p$ and $q$ assume the same truth value

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<th>$p \leftrightarrow q$</th>
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The truth table for the biconditional

Note: The biconditional statement $p \leftrightarrow q$ is often read as “$p$ if and only if $q$” or “$p$ is a necessary and sufficient condition for $q$.”
Truth tables can also be made for more complex expressions

**Example:** What is the truth table for \((p \land q) \to \neg r\)?

Subexpressions of “what we want to know”

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<tbody>
<tr>
<td>p</td>
<td>q</td>
<td>r</td>
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</table>

\(2^3 = 8\) rows

What we want to know

Like mathematical operators, logical operators are assigned precedence levels

1. Negation
   - \(\neg q \lor r\) means \((\neg q) \lor r\), not \(\neg (q \lor r)\)
2. Conjunction
3. Disjunction
   - \(q \land r \lor s\) means \((q \land r) \lor s\), not \(q \land (r \lor s)\)
4. Implication
   - \(q \land r \to s\) means \((q \land r) \to s\), not \(q \land (r \to s)\)
5. Biconditional

In general, we will try to use parenthesis to disambiguate these types of expressions.
Group Exercises

Problem 1: Show that an implication $p \rightarrow q$ and its contrapositive $\neg q \rightarrow \neg p$ always have the same value

- **Hint**: Construct two truth tables

Problem 2: Construct the truth table for the compound proposition $p \land (\neg q \lor r) \rightarrow s$

English sentences can often be translated into propositional sentences

But *why* would we do that?!?

- Philosophy and epistemology
- Reasoning about law
- Verifying complex system specifications
Example #1

Example: You can see an R-rated movie only if you are over 17 or you are accompanied by your legal guardian.

Let:

\[ r \equiv \text{"You can see an R-rated movie"} \]

\[ o \equiv \text{"You are over 17"} \]

\[ a \equiv \text{"You are accompanied by your legal guardian"} \]

Translation:

\[ r \rightarrow (o \lor a) \]

Example #2

Example: You can have free coffee if you are a senior citizen and it is a Tuesday.

Let:

\[ c \equiv \text{"You can have free coffee"} \]

\[ s \equiv \text{"You are a senior citizen"} \]

\[ t \equiv \text{"It is a Tuesday"} \]

Translation:

\[ (s \land t) \rightarrow c \]
Example #3

Example: If you are under 17 and are not accompanied by your legal guardian, then you cannot see the R-rated movie.

Let:

- \( r \equiv \text{"You can see the R-rated movie"} \)
- \( u \equiv \text{"You are under 17"} \)
- \( a \equiv \text{"You are accompanied by your legal guardian"} \)

Translation:

\[
(u \land \neg a) \rightarrow \neg r
\]

Note: The above translation is the contrapositive of the translation from example 1!

Logic also helps us understand bitwise operations

- Computers represent data as sequences of bits
  - E.g., 0101 1101 1010 1111

- Bitwise logical operations are often used to manipulate these data

- If we treat 1 as true and 0 as false, our logic truth tables tell us how to carry out bitwise logical operations
Bitwise logic examples

\[
\begin{array}{c}
\land \quad 1010 & 1110 \\
\lor \quad 1110 & 1010 \\
\land 1110 & 1010
\end{array}
\]

\[
\begin{array}{c}
\oplus 1010 & 1110 \\
\land \quad 1110 & 1010
\end{array}
\]

Group Exercises

**Problem 1:** Translate the following sentences

- If it is raining then I will either play video games or watch a movie
- You get a free salad only if you order off of the extended menu and it is a Wednesday

**Problem 2:** Solve the following bitwise problems

\[
\begin{array}{c}
\oplus 1011 & 1000 \\
\land \quad 1010 & 0110
\end{array}
\]

\[
\begin{array}{c}
\land 1011 & 1000 \\
\lor \quad 1010 & 0110
\end{array}
\]
Propositional logic is a simple logic that allows us to reason about a variety of concepts.