Information-gathering actions

- Some actions and their outcomes irreversibly change the world
- **Information-gathering (exploratory) actions:**
  - make an inquiry about the world
  - **Key benefit:** reduction in the uncertainty
- **Example: medicine**
  - Assume a patient is admitted to the hospital with some set of initial complaints
  - We are uncertain about the underlying problem and consider a surgery, or a medication to treat them
  - But there are often lab tests or observations that can help us to determine more closely the disease the patient suffers from
  - **Goal of lab tests:** Reduce the uncertainty of outcomes of treatments so that better treatment option can be chosen

Decision-making with exploratory actions

In decision trees:

- **Exploratory actions can be represented and reasoned about the same way as other actions.**

How do we capture the effect of exploratory actions in the decision tree model?

- Information obtained through exploratory actions may affect the probabilities of later outcomes
  - Recall that the probabilities on later outcomes can be conditioned on past observed outcomes and past actions
  - Sequence of past actions and outcomes is “remembered” within the decision tree branch
An oil wildcatter has to make a decision of whether to drill or not to drill on a specific site

- **Chance of hitting an oil deposit:**
  - Oil: 40% \( P(\text{Oil} = T) = 0.4 \)
  - No-oil: 60% \( P(\text{Oil} = F) = 0.6 \)

- **Cost of drilling:** 70K

- **Payoffs:**
  - Oil: 220K
  - No-oil: 0 K
Utility theory

Selection based on expected values

- **Until now:** The optimal action choice was the option that maximized the expected monetary value.
- **But is the expected monetary value always the quantity we want to optimize?**

![Diagram showing expected values for different assets and outcomes.](image_url)
Selection based on expected values

- Is the expected monetary value always the quantity we want to optimize?
- **Answer:** Yes, but only if we are risk-neutral.

- But what if we do not like the risk (we are risk-averse)?
- In that case we may want to get the premium for undertaking the risk (of loosing the money)
- **Example:**
  - we may prefer to get $101 for sure against $102 in expectation but with the risk of loosing the money
- **Problem:** How to model decisions and account for the risk?
- **Solution:** use utility function, and utility theory

Utility function

- **Utility function (denoted U)**
  - Quantifies how we “value” outcomes, i.e., it reflects our preferences
  - Can be also applied to “value” outcomes other than money and gains (e.g. utility of a patient being healthy, or ill)
- **Decision making:**
  - uses expected utilities (denoted EU)

\[
EU(X) = \sum_{x \in \Omega} P(X = x) U(X = x)
\]

\[
U(X = x) \quad \text{the utility of outcome } x
\]

**Important !!!**

- Under some conditions on preferences we can always design the utility function that fits our preferences
Utility theory

- Defines **axioms on preferences** that involve uncertainty and ways to manipulate them.
- Uncertainty is modeled through **lotteries**
  - **Lottery**:
    \[ [ p : A; (1 - p) : C] \]
    - Outcome A with probability p
    - Outcome C with probability (1-p)
- The following six constraints are known as the axioms of utility theory. The axioms are the most obvious semantic constraints on preferences with lotteries.

- **Notation**:
  - \( \succ \) - preferable
  - \( \sim \) - indifferent (equally preferable)

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Axioms of the utility theory

- **Orderability**: Given any two states, a rational agent prefers one of them, else the two as equally preferable.
  \[ (A \succ B) \lor (B \succ A) \lor (A \sim B) \]
- **Transitivity**: Given any three states, if an agent prefers A to B and prefers B to C, agent must prefer A to C.
  \[ (A \succ B) \land (B \succ C) \Rightarrow (A \succ C) \]
- **Continuity**: If some state B is between A and C in preference, then there is a \( p \) for which the rational agent will be indifferent between state B and the lottery in which A comes with probability p, C with probability (1-p).
  \[ (A \succ B \succ C) \Rightarrow \exists p [ p : A; (1 - p) : C] \sim B \]
Axioms of the utility theory

- **Substitutability**: If an agent is indifferent between two lotteries, \( A \) and \( B \), then there is a more complex lottery in which \( A \) can be substituted with \( B \).
  \[
  (A \sim B) \Rightarrow [p : A;(1 - p) : C] \sim [p : B;(1 - p) : C]
  \]

- **Monotonicity**: If an agent prefers \( A \) to \( B \), then the agent must prefer the lottery in which \( A \) occurs with a higher probability.
  \[
  (A \succ B) \Rightarrow (p > q \iff [p : A;(1 - p) : B] \succ [q : A;(1 - q) : B])
  \]

- **Decomposability**: Compound lotteries can be reduced to simpler lotteries using the laws of probability.
  \[
  [p : A;(1 - p) : [q : B;(1 - q) : C]] \Rightarrow
  [p : A;(1 - p)q : B;(1 - p)(1 - q) : C]
  \]

Utility theory

If the agent obeys the axioms of the utility theory, then

1. there exists a real valued function \( U \) such that:
   
   \[
   U(A) > U(B) \iff A \succ B
   \]
   \[
   U(A) = U(B) \iff A \sim B
   \]

2. The utility of the lottery is the expected utility, that is the sum of utilities of outcomes weighted by their probability
   
   \[
   U[p : A;(1 - p) : B] = pU(A) + (1 - p)U(B)
   \]

3. Rational agent makes the decisions in the presence of uncertainty by maximizing its expected utility
Utility functions

We can design a utility function that fits our preferences if they satisfy the axioms of utility theory.

- But how to design the utility function for monetary values so that they incorporate the risk?
- What is the relation between the utility function and monetary values?
- Assume we loose or gain $1000.
  - Typically this difference is more significant for lower values (around $100 - 1000) than for higher values (~ $1,000,000)
- What is the relation between utilities and monetary value for a typical person?

![Utility function graph](image-url)
Utility functions

- Expected utility of a sure outcome of 750 is 750

Assume a lottery L \([0.5: 500, 0.5:1000]\)
- Expected value of the lottery = 750
- Expected utility of the lottery EU(L) is different:
  - \(EU(L) = 0.5U(500) + 0.5*U(1000)\)
Utility functions

- Expected utility of the lottery $\text{EU}(\text{lottery } L) < \text{EU}(\text{sure } 750)$

- Risk aversion – a bonus is required for undertaking the risk

\[\text{Lottery } L: [0.5: 500, 0.5: 1000]\]