First-Order Logic

Chapter 8

Outline

• Why FOL?
• Syntax and semantics of FOL
• Using FOL
• Wumpus world in FOL
• Knowledge engineering in FOL
Pros and cons of propositional logic

😊 Propositional logic is declarative
😊 Propositional logic allows partial/disjunctive/negated information
  – (unlike most data structures and databases)
😊 Propositional logic is compositional:
  – meaning of $B_{1,1} \land P_{1,2}$ is derived from meaning of $B_{1,1}$ and of $P_{1,2}$
😊 Meaning in propositional logic is context-independent
  – (unlike natural language, where meaning depends on context)
😊 Propositional logic has very limited expressive power
  – (unlike natural language)
  – E.g., cannot say "pits cause breezes in adjacent squares"
    • except by writing one sentence for each square

First-order logic

• Whereas propositional logic assumes the world contains facts,
• first-order logic (like natural language) assumes the world contains
  – Objects: people, houses, numbers, colors, baseball games, wars, …
  – Relations: red, round, prime, brother of, bigger than, part of, comes between, …
  – Functions: father of, best friend, one more than, plus, …
FOL Syntax

• Add variables and quantifiers to propositional logic

Syntax of FOL: Basic elements

• Constants  KingJohn, 2, Pitt,...
• Predicates  Brother, >,...
• Functions  Sqrt, LeftLegOf,...
• Variables  x, y, a, b,...
• Connectives  ¬, →, ∧, ∨, ↔
• Equality  =
• Quantifiers  ∀, ∃
Atomic sentences

Atomic sentence = \textit{predicate} (term\textsubscript{1},...,term\textsubscript{n})
or term\textsubscript{1} = term\textsubscript{2}

Term = \textit{function} (term\textsubscript{1},...,term\textsubscript{n})
or constant or variable

• E.g., Brother(KingJohn,RichardTheLionheart)
• > (Length(LeftLegOf(Richard)),Length(LeftLegOf(KingJohn)))

Complex sentences

• Complex sentences are made from atomic sentences using connectives
  \neg S, \ S_1 \land S_2, \ S_1 \lor S_2, \ S_1 \Rightarrow S_2, \ S_1 \Leftrightarrow S_2

E.g. Sibling(KingJohn,Richard) \Rightarrow Sibling(Richard,KingJohn)

  >(1,2) \lor \leq (1,2)
  >(1,2) \land \neg >(1,2)
Knowledge engineering involves deciding what types of things should be constants, predicates, and functions for your problem.

### Propositional Logic vs FOL

\[ B_{33} \rightarrow (P_{32} \lor P_{23} \lor P_{34} \lor P_{43}) \ldots \]

“Internal squares adjacent to pits are breezy”:

\[ \text{All } X \; Y \; (B(X, Y) \land (X > 1) \land (Y > 1) \land (Y < 4) \land (X < 4)) \iff \]

\[ (P(X-1, Y) \lor P(X, Y-1) \lor P(X+1, Y) \lor (X, Y+1)) \]
FOL (FOPC) Worlds

- Rather than just T,F, now worlds contain:
- **Objects:** the gold, the wumpus, …
  “the domain”
- **Predicates:** holding, breezy
- **Functions:** sonOf

*Ontological commitment*

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Truth in first-order logic

- Sentences are true with respect to a **model** and an **interpretation**
- Model contains objects (**domain elements**) and relations among them
- Interpretation specifies referents for
  - constant symbols $\rightarrow$ objects
  - predicate symbols $\rightarrow$ relations
  - function symbols $\rightarrow$ functional relation

**Interpretation:** assignment of elements from the world to elements of the language

- An atomic sentence $\text{predicate}(\text{term}_1,\ldots,\text{term}_n)$ is true
  iff the objects referred to by $\text{term}_1,\ldots,\text{term}_n$
  are in the relation referred to by $\text{predicate}$
Models for FOL: Example

Quantifiers

- All $X \ p(X)$ means that $p$ holds for all elements in the domain
- Exists $X \ p(X)$ means that $p$ holds for at least one element of the domain
Universal quantification

Everyone at Pitt is smart:
\( \forall x \text{At}(x, \text{Pitt}) \Rightarrow \text{Smart}(x) \)

\( \forall x \ P \) is true in a model \( m \) iff \( P \) is true with \( x \) being each possible object in the model

Roughly speaking, equivalent to the conjunction of instantiations of \( P \):

\[
\text{At}(\text{KingJohn}, \text{Pitt}) \Rightarrow \text{Smart}(\text{KingJohn}) \\
\land \text{At}(\text{Richard}, \text{Pitt}) \Rightarrow \text{Smart}(\text{Richard}) \\
\land \text{At}(\text{Pitt}, \text{Pitt}) \Rightarrow \text{Smart}(\text{Pitt}) \\
\land \ldots
\]

A common mistake to avoid

Typically, \( \Rightarrow \) is the main connective with \( \forall \)

Common mistake: using \( \land \) as the main connective with \( \forall \):
\( \forall x \text{At}(x, \text{Pitt}) \land \text{Smart}(x) \)
means “Everyone is at Pitt and everyone is smart”
Existential quantification

- $\exists<\text{variables}> <\text{sentence}>$

- Someone at Pitt is smart:
  - $\exists x \text{At}(x,\text{Pitt}) \land \text{Smart}(x)$

- $\exists x P$ is true in a model $m$ iff $P$ is true with $x$ being some possible object in the model

- Roughly speaking, equivalent to the disjunction of instantiations of $P$
  - $\text{At(KingJohn, Pitt)} \land \text{Smart(KingJohn)}$
  - $\lor \text{At(Richard, Pitt)} \land \text{Smart(Richard)}$
  - $\lor \text{At(Pitt, Pitt)} \land \text{Smart(Pitt)}$
  - $\lor \ldots$

Another common mistake to avoid

- Typically, $\land$ is the main connective with $\exists$

- Common mistake: using $\Rightarrow$ as the main connective with $\exists$:
  - $\exists x \text{At}(x, \text{Pitt}) \Rightarrow \text{Smart}(x)$
  - is true if there is anyone who is not at Pitt!
Examples

• Everyone likes chocolate
• Someone likes chocolate
• Everyone likes chocolate unless they are allergic to it

Examples

• Everyone likes chocolate
  – \( \forall X \text{ person}(X) \rightarrow \text{likes}(X, \text{chocolate}) \)
• Someone likes chocolate
  – \( \exists X \text{ person}(X) \land \text{likes}(X, \text{chocolate}) \)
• Everyone likes chocolate unless they are allergic to it
  – \( \forall X (\text{person}(X) \land \neg \text{allergic}(X, \text{chocolate})) \rightarrow \text{likes}(X, \text{chocolate}) \)
Properties of quantifiers

• $\forall x \forall y$ is the same as $\forall y \forall x$
• $\exists x \exists y$ is the same as $\exists y \exists x$

• $\exists x \forall y$ is not the same as $\forall y \exists x$
• $\exists x \forall y$ Loves($x,y$)
  – “There is a person who loves everyone in the world”
• $\forall y \exists x$ Loves($x,y$)
  – “Everyone in the world is loved by at least one person”

Nesting of Variables

*Put quantifiers in front of* likes($P,F$)
*Assume the domain of discourse of $P$ is the set of people*
*Assume the domain of discourse of $F$ is the set of foods*

1. Everyone likes some kind of food
2. There is a kind of food that everyone likes
3. Someone likes all kinds of food
4. Every food has someone who likes it
Answers
(DOD of P is people and F is food)

Everyone likes some kind of food
\[ \text{All } P \text{ Exists } F \text{ likes}(P,F) \]

There is a kind of food that everyone likes
\[ \text{Exists } F \text{ All } P \text{ likes}(P,F) \]

Someone likes all kinds of food
\[ \text{Exists } P \text{ All } F \text{ likes}(P,F) \]

Every food has someone who likes it
\[ \text{All } F \text{ Exists } P \text{ likes}(P,F) \]

Answers, without Domain of Discourse Assumptions

Everyone likes some kind of food
\[ \text{All } P \text{ person}(P) \rightarrow \text{Exists } F \text{ food}(F) \text{ and likes}(P,F) \]

There is a kind of food that everyone likes
\[ \text{Exists } F \text{ food}(F) \text{ and (All } P \text{ person}(P) \rightarrow \text{likes}(P,F)) \]

Someone likes all kinds of food
\[ \text{Exists } P \text{ person}(P) \text{ and (All } F \text{ food}(F) \rightarrow \text{likes}(P,F)) \]

Every food has someone who likes it
\[ \text{All } F \text{ food } (F) \rightarrow \text{Exists } P \text{ person}(P) \text{ and likes}(P,F) \]
Quantification and Negation

- \(\neg (\forall x \ p(x)) \equiv \exists x \ \neg p(x)\)
- \(\neg (\exists x \ p(x)) \equiv \forall x \ \neg p(x)\)

- Quantifier duality: each can be expressed using the other
  - \(\forall x \ \text{Likes}(x,\text{IceCream}) \iff \neg \exists x \ \neg \text{Likes}(x,\text{IceCream})\)
  - \(\exists x \ \text{Likes}(x,\text{Broccoli}) \iff \neg \forall x \ \neg \text{Likes}(x,\text{Broccoli})\)

Equality

- \(\text{term}_1 = \text{term}_2\) is true under a given interpretation if and only if \(\text{term}_1\) and \(\text{term}_2\) refer to the same object

- E.g., definition of \(\text{Sibling}\) in terms of \(\text{Parent}\):
  \[\forall x,y \ \text{Sibling}(x,y) \iff [\neg (x = y) \land \exists m,f \ \neg (m = f) \land \text{Parent}(m,x) \land \text{Parent}(f,x) \land \text{Parent}(m,y) \land \text{Parent}(f,y)]\]
• Predicate of brotherhood: 
  – \{<R,J>,<J,R>\}
• Predicate of being on: \{<C,J>\}
• Predicate of being a person: 
  – \{J,R\}
• Predicate of being the king: \{J\}
• Predicate of being a crown: \{C\}
• Function for left legs: \{J(LL),R(LL)\}
Interpretation

• Specifies which objects, functions, and predicates are referred to by which constant symbols, function symbols, and predicate symbols.

• Under the intended interpretation:
  • “richardI” refers to R; “johnII” refers to J; “crown” refers to the crown.
  • “onHead”, “brother”, “person”, “king”, “crown”, “leftLeg”, “strong”

Lots of other possible interpretations

• 5 objects, so just for constants “richard” and “john” there are 25 possibilities
• Note that the legs don’t have their own names!
• “johnII” and “johnLackland” may be assigned the same object, J
• Also possible: “crown” and “johnII” refer to C (just not the intended interpretation)
Why isn’t the “intended interpretation” enough?

• Vague notion. What is intended may be ambiguous (and often is, for non-toy domains)
• Logically possible: square(x) ^ round(x). Your KB has to include knowledge that rules this out.

Determining truth values of FOPC sentences

• Assign meanings to terms:
  – “johnII” 🅐 J; “leftLeg(johnII)” 🅐 JLL
• Assign truth values to atomic sentences
  – “brother(johnII,richardI)”
  – “brother(johnlackland,richardI)”
  – Both True, because <J,R> is in the set assigned “brother”
  – “strong(leftLeg(johnlackland))”
  – True, because JLL is in the set assigned “strong”
Examples given the Sample Interpretation

- All $X,Y$ \(\text{brother}(X,Y)\) \text{FALSE}
- All $X,Y$ \((\text{person}(X) \land \text{person}(Y)) \rightarrow \text{brother}(X,Y)\) \text{FALSE}
- All $X,Y$ \((\text{person}(X) \land \text{person}(Y) \land \neg(X=Y)) \rightarrow \text{brother}(X,Y)\) \text{TRUE}
- Exists $X$ \text{crown}(X) \text{TRUE}
- Exists $X$ Exists $Y$ \text{sister}(X,Y) \text{FALSE}

Representational Schemes

- What are the objects, predicates, and functions? Keep in mind that you need to encode knowledge of specific problem instances and general knowledge.
- In practice, consider interpretations just to understand what the choices are. The world and interpretation are defined, or at least constrained, through the logical sentences we write.
Example Choice:
Predicates versus Constants

• Rep-Scheme 1: Let’s consider the world: $D = \{a,b,c,d,e\}$. green: $\{a,b,c\}$. blue: $\{d,e\}$. Some sentences that are satisfied by the intended interpretation:

  green(a).  green(b). blue(d).
  $\neg$(All x green(x)).  All x green(x) v blue(x).

But what if we want to say that blue is pretty?

Choice:
Predicates versus Constants

• Rep-Scheme 2: The world: $D = \{a,b,c,d,e,\text{green, blue}\}$
colorof:
  $\{\langle a, \text{green} \rangle, \langle b, \text{green} \rangle, \langle c, \text{green} \rangle, \langle d, \text{blue} \rangle, \langle e, \text{blue} \rangle\}$
pretty: $\{\text{blue}\}$  notprimary: $\{\text{green}\}$

• Some sentences that are satisfied by the intended interpretation:
  colorOf(a,green).  colorOf(b,green).  colorOf(d,blue).
  $\neg$(All X colorOf(X,green)).  All X colorOf(X,green) v colorOf(X,blue).
  ***pretty(blue).  notprimary(green).***

We have reified predicates blue and green: made them into objects.
Using FOL

The kinship domain:

- Brothers are siblings
  \[ \forall x,y \ Brother(x,y) \Rightarrow Sibling(x,y) \]

- One's mother is one's female parent
  \[ \forall m,c \ Mother(c) = m \Leftrightarrow (Female(m) \land Parent(m,c)) \]

- “Sibling” is symmetric
  \[ \forall x,y \ Sibling(x,y) \Leftrightarrow Sibling(y,x) \]

Interacting with FOL KBs

- Suppose a wumpus-world agent is using an FOL KB and perceives a smell and a breeze (but no glitter) at \( t=5 \):
  \[ \text{Tell}(KB, \text{Percept}([\text{Smell}, \text{Breeze}, \text{None}],5)) \]
  \[ \text{Ask}(KB, \exists a \text{ BestAction}(a,5)) \]

- I.e., does the KB entail some best action at \( t=5 \)?

- Answer: Yes, \{a/\text{Shoot}\} ← substitution (binding list)

- Given a sentence \( S \) and a substitution \( \sigma \),
  - \( S\sigma \) denotes the result of plugging \( \sigma \) into \( S \); e.g.,
    \[ S = \text{Smarter}(x,y) \]
    \[ \sigma = \{ x/\text{Hillary}, y/\text{Bill} \} \]
    \[ S\sigma = \text{Smarter}(\text{Hillary},\text{Bill}) \]

- \( \text{Ask}(KB,S) \) returns some/all \( \sigma \) such that \( KB \models S\sigma \)
Knowledge base for the wumpus world

• **Perception**
  – \( \forall t,s,b \ \text{Percept}([s,b,\text{Glitter}],t) \Rightarrow \text{Glitter}(t) \)

• **Reflex**
  – \( \forall t \ \text{Glitter}(t) \Rightarrow \text{BestAction(Grab,t)} \)

Deducing hidden properties

• \( \forall x,y,a,b \\text{Adjacent}([x,y],[a,b]) \Leftrightarrow [a,b] \in \{[x+1,y], [x-1,y],[x,y+1],[x,y-1]\} \)

Properties of squares:
• \( \forall s,t \ \text{At(Agent,s,t)} \land \text{Breeze(t)} \Rightarrow \text{Breezy(s)} \)

Squares are breezy near a pit:
  – **Diagnostic** rule---infer cause from effect
    \( \forall s \ \text{Breezy(s)} \Rightarrow \exists r \ \text{Adjacent}(r,s) \land \text{Pit(r)} \)
  – **Causal** rule---infer effect from cause
    \( \forall r \ \text{Pit(r)} \Rightarrow [\forall s \ \text{Adjacent}(r,s) \Rightarrow \text{Breezy(s)}] \)
Knowledge engineering in FOL

1. Identify the task
2. Assemble the relevant knowledge
3. Decide on a vocabulary of predicates, functions, and constants
4. Encode general knowledge about the domain
5. Encode a description of the specific problem instance
6. Pose queries to the inference procedure and get answers
7. Debug the knowledge base

Summary

• First-order logic:
  – objects and relations are semantic primitives
  – syntax: constants, functions, predicates, equality, quantifiers

• Increased expressive power: better to define wumpus world