CS 1674: Intro to Computer Vision
Geometric Transformations and Multiple Views

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Why multiple views?

• Structure and depth are inherently ambiguous from single views.

• Multiple views help us perceive 3d shape and depth.

Kristen Grauman, images from Svetlana Lazebnik
Alignment problem

- We previously discussed how to match features across images, of the same or different objects.
- Now let's focus on the case of “two images of the same object” (e.g. $x_i$ and $x_i'$).
- What transformation relates $x_i$ and $x_i'$?
- In alignment, we will fit the parameters of some transformation according to a set of matching feature pairs (“correspondences”).

Adapted from Kristen Grauman and Derek Hoiem
Motivation: Image mosaics
First, what are the correspondences?

- Compare content in **local** patches, find best matches.
  - Scan $x_i$’ with template formed from a point in $x_i$, and compute e.g. Euclidean distance between pixel intensities in the patch
  - Or compare SIFT features

Adapted from Kristen Grauman
Second, what are the transformations?

Examples of transformations:

- translate
- rotate
- change aspect ratio
- squish/shear
- change perspective

Adapted from Alyosha Efros
Parametric (global) warping

Transformation $T$ is a coordinate-changing machine:

$$ p' = T(p) $$

What does it mean that $T$ is **global**?

- It is the same for any point $p$
- It can be described by just a few numbers (parameters)

Let’s represent $T$ as a matrix:

$$ p' = Mp $$

$$ \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} M \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} $$
Scaling a coordinate means multiplying each of its components by a scalar.

Uniform scaling means this scalar is the same for all components.
Scaling

*Non-uniform scaling*: different scalars per component

Adapted from Alyosha Efros
Scaling

Scaling operation:

\[ x' = ax \]
\[ y' = by \]

Or, in matrix form:

\[
\begin{bmatrix}
  x' \\
  y'
\end{bmatrix}
= \begin{bmatrix}
  a & 0 \\
  0 & b
\end{bmatrix}
\begin{bmatrix}
  x \\
  y
\end{bmatrix}
\]

scaling matrix \( S \)

\[ x' = mx + ny \]
\[ y' = px + qy \]

Adapted from Alyosha Efros
2D Linear transformations

\[
\begin{bmatrix}
x' \\
y'
\end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}
\]

Only linear 2D transformations can be represented with a 2x2 matrix.

Linear transformations are combinations of …

- Scale,
- Rotation,
- Shear, and
- Mirror
What transforms can we write with 2x2 matrix?

2D Scaling?
\[
x' = s_x * x \\
y' = s_y * y
\]
\[
\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}
\]

2D Rotate around (0,0)? (see next slide)
\[
x' = \cos \Theta * x - \sin \Theta * y \\
y' = \sin \Theta * x + \cos \Theta * y
\]
\[
\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos \Theta & -\sin \Theta \\ \sin \Theta & \cos \Theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}
\]

2D Shear?
\[
x' = x + sh_x * y \\
y' = sh_y * x + y
\]
\[
\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & sh_x \\ sh_y & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}
\]
2D Rotation: Example

Θ = 90 \rightarrow M = [0 \ -1; 1 \ 0], i.e. x' = -y, y' = x

\[
\begin{bmatrix}
  x' \\
  y'
\end{bmatrix} = \begin{bmatrix}
  \cos \Theta & -\sin \Theta \\
  \sin \Theta & \cos \Theta
\end{bmatrix}\begin{bmatrix}
  x \\
  y
\end{bmatrix}
\]

X' = -Y,
Y' = X
2D Rotation: How to write

\[
\begin{bmatrix}
  x' \\
  y'
\end{bmatrix} = \begin{bmatrix}
  \cos \Theta & -\sin \Theta \\
  \sin \Theta & \cos \Theta 
\end{bmatrix} \begin{bmatrix}
  x \\
  y
\end{bmatrix}
\]

Polar coordinates…
\[
x = r \cos (\phi) \\
y = r \sin (\phi) \\
x' = r \cos (\phi + \theta) \\
y' = r \sin (\phi + \theta)
\]

Trig Identity…
\[
x' = r \cos(\phi) \cos(\theta) - r \sin(\phi) \sin(\theta) \\
y' = r \sin(\phi) \cos(\theta) + r \cos(\phi) \sin(\theta)
\]

\[
\begin{align*}
\sin(\alpha + \beta) &= \sin \alpha \cos \beta + \cos \alpha \sin \beta \\
\cos(\alpha + \beta) &= \cos \alpha \cos \beta - \sin \alpha \sin \beta \\
\sin(\alpha - \beta) &= \sin \alpha \cos \beta - \cos \alpha \sin \beta \\
\cos(\alpha - \beta) &= \cos \alpha \cos \beta + \sin \alpha \sin \beta
\end{align*}
\]

Substitute…
\[
x' = x \cos(\theta) - y \sin(\theta) \\
y' = x \sin(\theta) + y \cos(\theta)
\]

Adapted from Derek Hoiem, reference: [https://www2.clarku.edu/faculty/djoyce/trig/identities.html](https://www2.clarku.edu/faculty/djoyce/trig/identities.html)
What transforms can we write w/ 2x2 matrix?

2D Mirror about Y axis?

\[ x' = -x \]
\[ y' = y \]

\[
\begin{bmatrix}
  x' \\
  y'
\end{bmatrix} =
\begin{bmatrix}
  -1 & 0 \\
  0 & 1
\end{bmatrix}
\begin{bmatrix}
  x \\
  y
\end{bmatrix}
\]

2D Mirror over (0,0)?

\[ x' = -x \]
\[ y' = -y \]

\[
\begin{bmatrix}
  x' \\
  y'
\end{bmatrix} =
\begin{bmatrix}
  -1 & 0 \\
  0 & -1
\end{bmatrix}
\begin{bmatrix}
  x \\
  y
\end{bmatrix}
\]

2D Translation?

\[ x' = x + t_x \]
\[ y' = y + t_y \]

CAN’T DO!
Homogeneous coordinates

To convert to homogeneous coordinates:

\[(x, y) \Rightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}\]

homogeneous image coordinates

Converting \textit{from} homogeneous coordinates

\[
\begin{bmatrix} x \\ y \\ w \end{bmatrix} \Rightarrow (x/w, y/w)
\]

Simple example:

\[y = mx + b \quad \text{vs} \quad y = \mathbf{mx} \quad \text{where} \quad \mathbf{m} = [m \ b], \ \mathbf{x} = [x \ 1] \]
Translation

Homogeneous Coordinates

\[
\begin{bmatrix}
  x'' \\
  y'' \\
  1
\end{bmatrix} =
\begin{bmatrix}
  1 & 0 & t_x \\
  0 & 1 & t_y \\
  0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
  x \\
  y \\
  1
\end{bmatrix} =
\begin{bmatrix}
  x + tt_x \\
  y + tt_y \\
  1
\end{bmatrix}
\]

Adapted from Alyosha Efros
2D affine transformations

\[
\begin{bmatrix}
x' \\
y' \\
w'
\end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}
\]

Affine transformations are combinations of …

- Linear transformations, and
- Translations

Maps lines to lines, parallel lines remain parallel

Adapted from Alyosha Efros
Fitting an affine transformation

• Assuming we know the correspondences, how do we get the transformation?

\[
\begin{bmatrix}
  x'_i \\
  y'_i
\end{bmatrix} = \begin{bmatrix}
  m_1 & m_2 \\
  m_3 & m_4
\end{bmatrix} \begin{bmatrix}
  x_i \\
  y_i
\end{bmatrix} + \begin{bmatrix}
  t_1 \\
  t_2
\end{bmatrix}
\]

Alyosha Efros
Fitting an affine transformation

\[
\begin{bmatrix}
  x_i & y_i & 0 & 0 & 1 & 0 \\
  0 & 0 & x_i & y_i & 0 & 1 \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{bmatrix}
\begin{bmatrix}
  m_1 \\
  m_2 \\
  m_3 \\
  m_4 \\
  t_1 \\
  t_2 \\
\end{bmatrix} =
\begin{bmatrix}
  x'_i \\
  y'_i \\
  \vdots \\
\end{bmatrix}
\]

- How many matches (correspondence pairs) do we need to solve for the transformation parameters?
- Once we have solved for the parameters, how do we compute \((x'_{new}, y'_{new})\) given \((x_{new}, y_{new})\)?
Detour: Keypoint matching for search

1. Find a set of distinctive keypoints
2. Define a region around each keypoint (window)
3. Compute a local descriptor from the region
4. Match descriptors

\[ d(f_A, f_B) < T \]

Adapted from K. Grauman, B. Leibe
Detour: solving for translation with outliers

Given matched points in $\{A\}$ and $\{B\}$, estimate the translation of the object

$$
\begin{bmatrix}
  x_i^B \\
  y_i^B
\end{bmatrix} =
\begin{bmatrix}
  x_i^A \\
  y_i^A
\end{bmatrix} +
\begin{bmatrix}
  t_x \\
  t_y
\end{bmatrix}
$$
Detour: solving for translation with outliers

Problem: outliers

Hough transform solution

1. Initialize a grid of parameter values
2. Each matched pair casts a vote for consistent values
3. Find the parameters with the most votes

Adapted from Derek Hoiem
Detour: solving for translation with outliers

Problem: multiple objects

Hough transform solution

1. Initialize a grid of parameter values
2. Each matched pair casts a vote for consistent values
3. Find the parameters with the most votes
2D projective transformations

\[
\begin{bmatrix}
x' \\
y' \\
w'
\end{bmatrix} = \begin{bmatrix}
a & b & c \\
d & e & f \\
g & h & i \\
\end{bmatrix} \begin{bmatrix}
x \\
y \\
w
\end{bmatrix}
\]

Projective transformations:
- Affine transformations, and
- Projective warps

Parallel lines do not necessarily remain parallel
Projective transformations

A projective transformation is a mapping between any two projective planes with *the same center of projection*

Also called **Homography**

\[
\begin{bmatrix}
wx' \\
w y'
\end{bmatrix} = \begin{bmatrix}
* & * & * \\
* & * & * \\
* & * & *
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
l
\end{bmatrix}
\]

Adapted from Alyosha Efros
Image mosaics: Camera setup

Two images with camera rotation but no translation

Adapted from Derek Hoiem
Image mosaics: Goals

Obtain a wider angle view by combining multiple images.
Image mosaics: Many 2D views, one 3D object

The mosaic has a natural interpretation in 3D
- The images are reprojected onto a common plane
- The mosaic is formed on this plane
- Mosaic is a synthetic wide-angle camera
How to stitch together panorama (mosaic)?

Basic Procedure

• Take a sequence of images from the same position
  – Rotate the camera about its optical center

• **Compute the homography** (transformation) between first and second image

• **Combine images** (draw first image onto second’s canvas)

• Blend the two together to create a mosaic (post-process)

• (If there are more images, repeat)

Adapted from Steve Seitz
To compute the homography given pairs of corresponding points in the images, we need to set up an equation where the parameters of $H$ are the unknowns...
Computing the homography

• Assume we have four matched points: How do we compute homography \( H \)?

\[
p' = Hp
\]

\[
p' = \begin{bmatrix} wx' \\ wy' \\ w \end{bmatrix}
\]

\[
H = \begin{bmatrix} h_1 & h_2 & h_3 \\ h_4 & h_5 & h_6 \\ h_7 & h_8 & h_9 \end{bmatrix}
\]

\[
p = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}
\]

\[
h = \begin{bmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \\ h_5 \\ h_6 \\ h_7 \\ h_8 \\ h_9 \end{bmatrix}
\]

Can set scale factor \( h_9 = 1 \).
So, there are 8 unknowns.
Need at least 8 eqs, but the more the better…

\[
A \begin{bmatrix} -x & -y & -1 & 0 & 0 & 0 & xx' & yy' & x' \ 0 & 0 & 0 & -x & -y & -1 & xy' & yy' & y' \ ... \end{bmatrix} h = 0
\]

Adapted from Derek Hoiem, Kristen Grauman

How to stitch together panorama (mosaic)?

**Basic Procedure**

- Take a sequence of images from the same position
  - Rotate the camera about its optical center
- **Compute the homography** (transformation) between first and second image
- **Combine images** *(draw first image onto second’s canvas)*
- Blend the two together to create a mosaic *(post-process)*
- *(If there are more images, repeat)*

Adapted from Steve Seitz
Combining images

To apply a given homography $H$

- Compute $p' = Hp$ (regular matrix multiply)
- Convert $p'$ from homogeneous to image coordinates

$\begin{bmatrix} wx' \\ wy' \\ w \end{bmatrix} = \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$
Combining images

Forward warping:
Send each pixel \( f(x,y) \) to its corresponding location \((x',y') = H(x,y)\) in the right image.

Modified from Alyosha Efros
Combining images

**Forward warping:**
Send each pixel $f(x,y)$ to its corresponding location 
$$(x',y') = H(x,y)$$ in the right image

Q: what if pixel lands “between” two pixels?
A: *round* values of $(x',y')$ or *distribute* color among neighbors

Adapted from Alyosha Efros
Combining images

Inverse warping:
Get each pixel $g(x', y')$ from its corresponding location $(x, y) = H^{-1}(x', y')$ in the left image
Combining images

Inverse warping:
Get each pixel $g(x', y')$ from its corresponding location $(x, y) = H^{-1}(x', y')$ in the left image

Q: what if pixel comes from “between” two pixels?
A: *interpolate* color value from neighbors
Homography example: Image rectification

To unwarp (rectify) an image solve for homography $H$ given $p$ and $p'$: $p' = Hp$
Summary of affine/projective transforms

- **Write 2d transformations** as matrix-vector multiplication (including translation when we use homogeneous coordinates)

- **Fitting transformations**: solve for unknown parameters given corresponding points from two views – linear, affine, projective (homography)

- **Mosaics**: uses homography and image warping to merge views taken from same center of projection
  - Perform **image warping** (forward, inverse)

Adapted from Kristen Grauman
Next: Stereo vision

- Homography: Same camera center, but camera rotates
- Stereo vision: Camera center is not the same (we have multiple cameras)

- Epipolar geometry
  - Relates cameras from two positions/cameras

- Stereo depth estimation
  - Recover depth from disparities between two images

Adapted from Derek Hoiem
Stereo photography and stereo viewers

Take two pictures of the same subject from two slightly different viewpoints and display so that each eye sees only one of the images.

Invented by Sir Charles Wheatstone, 1838

Image from fisher-price.com
Depth from stereo for computers

Two cameras, simultaneous views

Single moving camera and static scene
Depth from stereo

- Goal: recover depth by finding image coordinate $x'$ that corresponds to $x$
Depth from stereo

- Goal: recover depth by finding image coordinate $x'$ that corresponds to $x$

- Sub-Problems
  
  1. Calibration: How do we recover the relation of the cameras (if not already known)?
  
  2. Correspondence: How do we search for the matching point $x'$?

  3. Estimate depth from matches
Geometry for a simple stereo system

- Assume parallel optical axes, known camera parameters (i.e., calibrated cameras). **What is expression for Z?**

\[ T + x_l - x_r \]
\[ Z - f \]
\[ Z \]

**Similar triangles** \((p_l, P, p_r)\) and \((O_l, P, O_r)\):

\[ \frac{T + x_l - x_r}{Z - f} = \frac{T}{Z} \]

- Depth is inversely proportional to disparity.

Adapted from Kristen Grauman
Depth from disparity

- We have two images from different cameras.
- If we could find the **corresponding points** in two images, we could estimate relative depth…
- How do we match a point in the first image to a point in the second **efficiently**?

image $l(x,y)$  
Disparity map $D(x,y)$  
image $l'(x',y')$
Stereo correspondence constraints

• Given $p$ in left image, where can corresponding point $p'$ be?
• **Baseline** – line connecting the two camera centers

• **Epipoles**
  = intersections of baseline with image planes
  = projections of the other camera center

• **Epipolar Plane** – plane containing baseline

• **Epipolar Lines** - intersections of epipolar plane with image planes (always come in corresponding pairs)
Epipolar constraint

Geometry of two views constrains where the corresponding pixel for some image point in the first view must occur in the second view.

- It must be on the line where (1) the plane connecting the world point and optical centers, and (2) the image plane, intersect.
- Potential matches for $p$ have to lie on the corresponding line $l'$.
- Potential matches for $p'$ have to lie on the corresponding line $l$.

Adapted from Kristen Grauman, Derek Hoiem
The epipolar constraint is useful because it reduces the correspondence problem to a 1D search along an epipolar line.
If the stereo rig is calibrated, we know how to rotate and translate camera reference frame 1 to get to camera reference frame 2.

- Rotation: 3x3 matrix $R$; translation: 3x1 vector $T$.

\[ X' = RX + T \]

(See hidden slides for how we get to the next slide.)

Adapted from Kristen Grauman
Essential matrix

\[
X' \cdot (T \times RX) = 0
\]

\[
X' \cdot ([T_x]RX) = 0
\]

Let \( E = [T_x]R \)

\[
X' \cdot EX = X'^T EX = 0
\]

\( E \) is called the essential matrix, and it relates corresponding image points between both cameras, given the rotation and translation.

Before we said: If we observe a point in one image, its position in other image is constrained to lie on line defined by above. It turns out that:

- \( E^T x \) is the epipolar line \( l' \) through \( x' \) in the second image, corresponding to \( x \).
- \( Ex' \) is the epipolar line \( l \) through \( x \) in the first image, corresponding to \( x' \).

Adapted from Kristen Grauman, Derek Hoiem
Essential matrix example: parallel cameras

For the parallel cameras, image of any point must lie on same horizontal line in each image plane.

\[
\begin{align*}
R &= \\
T &= \\
E &= [T_x]R = \\
p &= [x, y, f] \\
p' &= [x', y', f] \\
p'^T E p &= 0
\end{align*}
\]
image $I(x,y)$  

Disparity map $D(x,y)$  

image $I'(x',y')$  

$(x',y') = (x + D(x,y), y)$
Basic stereo matching algorithm

• For each pixel in the first image
  – Find corresponding epipolar scanline in the right image
  – Search along epipolar line and pick the best match $x'$: slide a window along the right scanline and compute Euclidean distance between contents of that window with the reference window in the left image; take the window corresponding to the minimum as the match
  – Compute disparity $x-x'$ and set $\text{depth}(x) = fT/(x-x')$

Adapted from Derek Hoiem
Results with window search

Data

Predicted depth

Ground truth
Summary of stereo vision

• **Epipolar geometry**
  – Epipoles are intersection of baseline with image planes
  – Matching point in second image is on a line passing through its epipole
  – Epipolar constraint limits where points from one view will be imaged in the other, which makes search for correspondences quicker
  – Essential matrix E maps from a point in one image to a line (its epipolar line) in the other

• **Stereo depth estimation**
  – Find corresponding points along epipolar scanline
  – Estimate disparity (depth is inverse to disparity)

Adapted from Kristen Grauman and Derek Hoiem
Projective structure from motion

• Given: $m$ images of $n$ fixed 3D points
  \[ x_{ij} = P_i X_j, \quad i = 1, \ldots, m, \quad j = 1, \ldots, n \]

• Problem: estimate $m$ projection matrices $P_i$ and $n$ 3D points $X_j$ from the $mn$ corresponding 2D points $x_{ij}$
Photo tourism


http://phototour.cs.washington.edu/
3D from multiple images