LP-based Approximation Algorithms for Broadcast Scheduling

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Dependent rounding in bipartite graphs. Proc. FOCS’02
Offline Broadcast Scheduling [ESA00]

- Broadcast server with slotted broadcast channel
- $n$ one-slot pages $P_1, P_2, \ldots, P_n$
- Requests $R_i(t), 1 \leq i \leq n$ are received at the end of slot $t$
- Objective is to minimize avg. (total) response time

\[
\begin{array}{cccccc}
\hline
& t & t+1 & t+2 & t+3 & t+4 & t+5 \\
\hline
R_i(t) & c_i(t) & P_i \\
\hline
\end{array}
\]
Resource Augmentation

- Offline problem shown NP-hard

- Ultimate goal: $O(1)$ approximation algorithm

- Resource augmentation [ESA00]: Can we get an $O(1)$ approximation alg. by throwing $O(1)$ more bandwidth at the problem?

- A $k$-speed server is one that broadcasts up to $k$ pages every time slot.
Summary of results

- **ESA00**: 3-speed 3-approx. alg.
- **IPCO02**: 2-speed 2-approx and 4-speed 1-approx
- **FOCS02**: 2-speed 2-approx and 3-speed 1-approx
**ESA00: Strategy overview**

- Reduction to integer min-cost flow reduction
- Probabilistic interpretation of the LP solution
- Impose a structure on the LP solution to facilitate the definition of a $1/\alpha$-speed alg. ($\alpha < 1/2$ and $1/\alpha$ is an integer)
- Prove that the algorithm guarantees a factor-$\frac{1}{1-2\alpha}$ approximate solution
Reduction to Min-cost flow

\[
\begin{align*}
  \text{Page a} & : v_b(0) \\
  \text{Page b} & : v_b(1) \\
  \text{Page c} & : v_b(2) \\
  \text{Page d} & : v_b(3) \\
  \text{Page e} & : v_b(4)
\end{align*}
\]
Min-cost flow ILP

\[
\min \sum_{t=0}^{T+n} \sum_{t'=t+1}^{T+n} \sum_{i=1}^{n} w_i(t, t') f_i(t, t')
\]

(1)

\[
\sum_{i=1}^{n} f_i(-1, 0) = n
\]

(2)

\[
\sum_{t'<t} f_i(t', t) = \sum_{t'>t} f_i(t, t') \quad 1 \leq i \leq n \quad 0 \leq t \leq T+n
\]

(3)

\[
\sum_{i=1}^{n} \sum_{t'<t} f_i(t', t) \leq 1 \quad 1 \leq t \leq T+n
\]

(4)

\[
f_i(t, t') \in \{0, 1\} \quad 1 \leq i \leq n \quad 0 \leq t < t' \leq T+n
\]

(5)

\[
f_i(t, n+t+1) \in \{0, 1\} \quad 1 \leq i \leq n \quad T+1 \leq t \leq T+n
\]

(6)

Lemma 1. The optimal value of IP is the minimum total response time for the broadcast problem.
Fractional LP solution

- LOPT is \( n \)-feasible: fraction of each page can be transmitted in a slot without violating the fourth constraint.

**Lemma 2.** All requests are eventually satisfied in the LP solution, equiv.,
\[
R_i(t) > 0 \implies \sum_{0 \leq j \leq k \leq T+n} f_i(j,k) = 1
\]

- Probabilistic interpretation of LP solution

\[
\sum_{t' < t} f_i(t',t) \equiv p_i(t) \quad \Pr\{P_i \text{ is broadcast at } t\}
\]

\[
\min\{t' > t : p_i(t') \geq 1\} \equiv n_i(t) \quad \text{Requests } (i,t) \text{ are fulfilled by } n_i(t)
\]

\[
1 - \sum_{j=t}^{n_i(t)-1} p_i(t) \equiv \ell_i(t) \quad \Pr\{\text{Requests } (i,t) \text{ are satisfied at } n_i(t)\}
\]
Probabilistic interpretation

- The $R_i(t) > 0$ requests are satisfied at $t < s < n$ with prob. $p_i(s)$
- They are satisfied at $n_i(t)$ with prob. $\ell_i(t)$
- The prob. solution

$$VL = \sum_{i=1}^{n} \sum_{t=1}^{T} R_i(t) \left[ \sum_{t'=t+1}^{n_i(t)-1} [p_i(t')(t' - t)] + \ell_i(t)(n_i(t) - t) \right]$$

**Lemma 3.** $VL = LOPT$
Restructuring the fractional solution

• Let $\alpha < 1/2$, $1/\alpha$ is an integer

• They seek a $1/\alpha$-feasible solution

• $b_i(j) = \min\{t : \sum_{k=1}^{t} p_i(k) \geq j\alpha\}$, the $j^{th}$ breakpoint for $P_i$

• $I = \{I_i(j) = [b_i(j), b_i(j+1)] | 1 \leq i \leq n, 1 \leq j \leq h(i)\}$

• Interval $I_i(j)$ is satisfied (at time $t$) if $P_i$ is broadcast at $t \in I_i(j)$
The LEDF algorithm

- Assume all requests up to time $t$ have been scheduled

- To schedule the requests at $t$: let $U$ be the set of currently unsatisfied intervals and
  1. Pick from $U$ up to $1/\alpha$ intervals with smallest right endpoints
  2. broadcast them at time $t$

- **Lemma 4.** *Every interval is satisfied in LEDF*
LEDF is a $\frac{1}{1-2\alpha}$ approximation alg.

- **Lemma 5.** The total response time of the $1/\alpha$-feasible schedule LEDF is at most $\frac{LOPT}{1-2\alpha}$

- **Theorem 1.** There is a polynomial-time 3-speed 3-approx. alg. for the broadcast problem where the objective is to min. the average response time.
IPCO02: Strategy overview

• ILP formulation

• Relaxed LP solution

• Using LOPT, define a simplified problem instance that allows a $1/\alpha$-speed alg. ($\alpha \leq 1/2$ and $1/\alpha$ is an integer).

• The simplified instance can be reduced to min-cost flow on bipartite graph.

• Exact solution to relaxed simplified instance by the integrality theorem.

• Prove the solution for simplified instance is at most $\frac{1}{1-\alpha}$ the optimal solution.
ILP formulation

• $y_{i,t'} = 1$ iff. $P_i$ is broadcast at $t'$
• $x_{i,t,t'} = 1$ iff. requests $(i,t)$ are satisfied at $t' > t$

\[
\min \sum_{t=0}^{T} \sum_{t'=t+1}^{T+n} \sum_{i=1}^{n} (t' - t) R_i(t) x_{i,t,t'}
\]

(7)

\[
y_{i,t'} - x_{i,t,t'} \geq 0 \quad 1 \leq i \leq n, \quad t' > t
\]

(8)

\[
\sum_{t'=t+1}^{T+n} x_{i,t,t'} \geq 1 \quad 1 \leq i \leq n, \quad \forall t
\]

(9)

\[
\sum_{i} y_{i,t'} \leq 1 \quad \forall t'
\]

(10)

\[
x_{i,t,t'} \in \{0, 1\} \quad 1 \leq i \leq n, \quad \forall t, t'
\]

(11)

\[
y_{i,t'} \in \{0, 1\} \quad 1 \leq i \leq n, \quad \forall t'
\]

(12)
The simplified problem instance

- Let \((x, y)\) be the optimal fractional solution

- For \(R_i(t) > 0\), define \(f_t(\alpha, i, t) = \min\{t''|\sum_{t' = t+1}^{t''} x_{it'} \geq \alpha\}\)

- Transform \(I\) to \(I'\) by consolidating the requests so that the following property holds for \(I'\):

  **Property 1.** Let \(N_i'\) be the set times of requests for \(P_i\) in \(I'\), and \(t_u' < t_v'\) then, \(\{t, t'\} \subseteq N_i' \implies f_t(\alpha, i, t_u') \leq t_v'

- Requests \((i, t)\) grouped with \((i, g(i, t))\), \(g(i, t) \geq t\) \(\implies f_t(\alpha, i, t) > g(i, t)\)
$1/\alpha$ fractional solution for $I'$

- Recall $(x, y)$ is the optimal $n$-feasible fractional solution for $I$

- Consider the solution $(x_\alpha, y)$

\[
x^i_{\alpha t'} = \begin{cases} 
  x^i_{tt'} & \text{if } t' < ft(\alpha, i, t) \\
  \alpha - \sum_{t''=t+1}^{t'-1} x^i_{tt''} & \text{if } t' = ft(\alpha, i, t) \\
  0 & \text{otherwise}
\end{cases}
\]

- The solution $(\frac{1}{\alpha}x_\alpha, \frac{1}{\alpha}y)$ is $1/\alpha$-feasible in the weak sense that constraint becomes $\sum_i y^i_{t'} \leq \alpha, \forall t'$.

However, without integer solution $(x_\alpha, y)$, up to $n$ pages may need to be broadcast each time slot.
\[\frac{1}{\alpha}\ \text{integer solution for } I'\]

- Construct the following min. cost flow network \(N\)
Analysis

- **Lemma 6.** The cost of $1/\alpha$-speed frac. sol. in $I'$ and that of an optimal frac. sol. for $I$ are related as follows

\[
\frac{1}{\alpha} \sum_{(i,t) \in I} R_i(t) \left[ \sum_{t = g(i,t) + 1}^{T + n} (t' - g(i,t)) x_{\alpha g(i,t)t'} \right] \leq \frac{1}{1 - \alpha} \sum_{(i,t) \in I} R_i(t) (g(i,t) - t)
\]

- **Lemma 7.** For any feasible flow in $N$ there is a $1/\alpha$-speed feasible fractional sol. for $I'$ of the same cost, and vice-versa

- **Lemma 8.** There exists a $1/\alpha$-speed integral solution for $I'$ of the same cost as the $1/\alpha$-speed fractional sol. $(\frac{1}{\alpha}x_{\alpha}, \frac{1}{\alpha}y)$

- **Theorem 2.** There is a $1/\alpha$-speed, $\frac{1}{1-\alpha}$ approx. solution for the Broadcast Problem.
FOCS02: Strategy overview

- Same ILP formulation as IPCO02
- Same relaxed LP solution
- Using LOPT, define a simplified problem instance that allows a 1/2-speed alg. Difference: don’t combine requests, just drop those in the middle.
- The simplified instance can be reduced to min-cost flow on bipartite graph. Different from IPCO02.
- Apply Dependent rounding on the bipartite graph to get a 1/2-speed 2-approx. integral sol.
- A 3-speed, 1-approx. algorithm
Dependent Rounding (DR)

- Bipartite graph \((A, B, E)\) with bipartition \((A, B)\)
- \(x_{ij} \in [0, 1]\) for each \((i, j) \in E\).
- DR is a poly-time randomized scheme that rounds each \(x_{i,j}\) to \(X_{i,j} \in \{0, 1\}\) such that the following properties hold

**Property 2. Marginal distribution.** \(\Pr[X_{i,j} = 1] = x_{i,j}\)

**Property 3. Degree preservation.** Define fractional degree \(d_i = \sum_{j:(i,j) \in E} x_{i,j}\), and integral degree \(D_i = \sum_{j:(i,j) \in E} X_{i,j}\), then
\[
\Pr[D_i \in \{\lfloor d_i \rfloor, \lceil d_i \rceil\}] = 1
\]

**Property 4. Negative correlation.** Let \(f\) be an edge in \(E\), \(\forall b \in \{0, 1\}\), \(\Pr[\bigwedge_{f \in S}(X_f = b)] \leq \prod_{f \in S} \Pr[X_f = b]\)