1. We consider two different languages that in some sense represent strings representing correct additive equalities. You can assume that the input is on a one-way read-only tape. Show that one of these problems can be solved by a finite state machine. Show from first principles (so no using results like the Pumping Lemma) that one of these problems can not be solved by a finite state machine. To show impossibility, you should assume that there is a finite automata $M$ with $k$ states that accepts this language. Then consider arbitrarily large strings of some particular type, and argue $M$ has to mess up on some string of length greater than $k$.

(a) The language consists of strings over the symbols: 0, 1, + and =. The languages consists of strings of the form $x + y = z$, where natural numbers $x, y$ and $z$ encoded in binary in the standard way, and where $x$ plus $y$ is indeed equal to $z$. So for example the string 10100 + 111 = 11011 of 11 symbols is in the language.

(b) The language is a string of the following 8 symbols (so each line is a different symbol)

```
[000]
[001]
[010]
[011]
[100]
[101]
[110]
[111]
```

where the first column plus the second column is equal to the third column. So to reuse our example from the previous subproblem, the 5 symbol input

```
[101][001][110][011][011]
```

is in the language because 10100 + 111 = 11011.

So this shows that whether one whether a finite state machine can add, depends on the definition of notion of addition that you use.

Due Friday January 12 (You need not use LaTeX, your solutions may be handwritten.)

2. Problem 1.1 from the text. That is, construct a complete formal description of a Turing machine (using the formalization given in section 1.2) that adds two binary numbers (you need not do multiplication). Assume that input is of the form NUMBER1#NUMBER2, e.g. 1011#110. You should write your output on a second tape.

Due Friday January 12 (You need not use LaTeX, your solutions may be handwritten.)

3. Read the description of the Post Correspondence Problem

https://en.wikipedia.org/wiki/Post_correspondence_problem
including the sketch of the proof that there is no algorithm to solve this problem. The article ends with "There are a number of details to work out, such as dealing with boundaries between states, making sure that our initial tile goes first in the match, and so on, .."

Explain how to deal with these two details, that is, how to deal with boundaries between states, and how to make sure that the initial tile goes first. List any other details that you think need to be dealt with, and explain how they can be dealt with.

Due Wednesday January 17.

4. (a) (warm up) Show by reduction from the Halting Problem that there is no Turing machine that takes as input a Turing machine $M$, and determines whether the language $L(M)$ accepted by $M$ is the empty language.

(b) (warm up) Show by reduction from the Halting Problem that there is no Turing machine that takes as input a Turing machine $M$, and determines whether the language $L(M)$ accepted by $M$ is the language of every string over the input alphabet.

(c) (warm up) Show by reduction from the Halting Problem that there is no Turing machine that takes as input a Turing machine $M$, and determines whether the language $L(M)$ accepted by $M$ includes the string 11110.

(d) Let $P$ be some property of languages. Further assume there is a Turing machine $M_1$ that accepts a language $L_1$ that has property $P$, and a Turing machine $M_2$ that accepts a language $L_2$ that does not have has property $P$. Show by reduction from the Halting Problem that there is no Turing machine that takes as input a Turing machine $M$, and determines whether the language $L(M)$ accepted by $M$ satisfies property $P$.

(e) Explain why the first three subproblems are consequences of the fourth subproblem.

Due Friday January 19. Note that some students may find the fourth subproblem somewhat tricky. Do the other subproblems first, and then make a good faith effort at the fourth subproblem. At the very least, at least understand the statement of the fourth subproblem.

5. Show that the following two definitions of recursively enumerable are logically equivalent:

(a) A language $L$ is recursively enumerable iff there is a Turing machine $M$ such that if $x \in L$ then $M$ accepts $x$ and if $x \notin L$ then $M$ loops forever on $x$.

(b) A language $L$ is recursively enumerable iff there is a Turing machine $M$, with a read/write tape that is initially empty and a write-only output tape, such that only elements of $L$ are written to the output tape, and every element of $L$ is eventually written to the output tape.

Due Monday January 22
6. Consider a proof rule such as:

**Proof Rule:** From the statement \( \forall x \ P(x) \) one can deduce the countably infinite number of statements \( P(0), P(1), P(2), P(3), \) etc.

Somewhat more formally, you can assume that there is a Turing machine \( T \) that takes as input a statement \( S \), and will output a list all the statements one can conclude from \( S \) using this proof rule. Here “list” means what it means within the context of the definition of recursively enumerable, \( T \) will only output statements we can deduce from \( S \) by the proof rule, and every statement that we can deduce from \( S \) by the proof rule will eventually be listed. Prove that the following language is still recursively enumerable:

\[
L = \{ S \mid \text{statement } S \text{ is provable from the finite set } A \text{ of axioms} \}
\]

Hint: Now the nodes in the tree of all proofs can have infinitely many children. So breadth first search will no longer work. So you will have to find another way to search this tree so that every node is eventually reached.

Due Monday January 22

7. (a) Consider the following one tape Turing Machine \( M \) that halts iff the number of 1’s on the input tape is odd.

- **tape alphabet = \{0, 1, space\}**
- **start state = q_0**
- **states = \( Q = \{ q_0, q_1, q_h \} \)**
- **halt state = q_h. So the machine stops running if it ever reaches state q_h.**

Transitions:

<table>
<thead>
<tr>
<th>State</th>
<th>Symbol</th>
<th>Action</th>
</tr>
</thead>
<tbody>
<tr>
<td>( q_0 )</td>
<td>0</td>
<td>( q_0 ), 0, right</td>
</tr>
<tr>
<td>( q_0 )</td>
<td>1</td>
<td>( q_1 ), 1, right</td>
</tr>
<tr>
<td>( q_0 )</td>
<td>space</td>
<td>( q_0 ), space, stay</td>
</tr>
<tr>
<td>( q_1 )</td>
<td>0</td>
<td>( q_1 ), 1, right</td>
</tr>
<tr>
<td>( q_1 )</td>
<td>1</td>
<td>( q_0 ), 0, right</td>
</tr>
<tr>
<td>( q_1 )</td>
<td>space</td>
<td>( q_h ), space, stay</td>
</tr>
</tbody>
</table>

Show one valid computation history \( H \) for the input \( I = 101 \) for Turing machine \( M \). Note that the computation history will have the form \#C_0\#C_1\#C_2\#...\#C_k\# where \( k \) is the number of steps that \( M \) runs on \( I \) and \( C_i \) is the configuration of \( M \) on \( I \) after \( i \) steps. Note that there technically there is more than one valid computation history depending on how many spaces at the right of the tape are included in each configuration.

(b) Give the Godel sentence \( S \) for the \( M \) and \( I \) in the previous subproblem. That is, \( S \) should be a first order sentence in the language of number theory that will be true iff and only if \( M \) halts on \( I \).

Hint: To get you started, \( S \) should ask whether there exists a number \( H \), which one can interpret as a computation history of \( M \) on \( I \). Conditions then need
to be added to \( S \) to to check that \( H \) is a valid computation history of \( M \) on \( I \) that shows that \( M \) halts. You are strongly encouraged to use macros (see http://www.computerhope.com/jargon/m/macro.htm ), which of course you have to define.

Some macros that might be useful are:

- \( \text{PLACE}(j) \) represents an arithmetic expression that returns the digit in location \( j \) in \( H \).
- \( \text{SAME}(i, j, k, l) \) represents a logical expression that will be true iff digits \( i \) through \( j \) are identical to digits \( k \) through \( l \).
- \( \text{STATE}(i) \) represents a local expression that will be true iff only digit \( i \) of \( H \) represents a state in \( Q \).
- \( \text{TABLE}(i, j) \) represents a local expression that will be true iff digits \( i \), \( i+1 \) and \( i+2 \) in \( H \) represent a tape symbol, state in \( Q \) and a tape symbol respectively, and digits \( j \), \( j+1 \) and \( j+2 \) in \( H \) evolve properly from digits \( i \), \( i+1 \) and \( i+2 \) according to \( M \). So for example if digits \( i \), \( i+1 \) and \( i+2 \) were 0, \( q_0 \), 1 respectively, then \( \text{TABLE}(i, j) \) would only be true iff digits \( j \), \( j+1 \), \( j+2 \) were 0, 1, \( q_1 \).

You are welcome to use other macros.

Due Wednesday January 24

8. Consider first order logical sentences of arithmetic where you are only allowed to use the arithmetic operators \( = \) and \( + \). Without loss of generality we may assume that the only logical operators are AND, represented by \( \land \) and NOT, represented by \( \neg \), and that all quantifiers appear first. So you might have a formula like:

\[
\forall x\exists y\forall z\exists w \ (x + y = z) \land \neg(y + z = w + x)
\]

Our goal here is to show that there is an algorithm to accept exactly the language of true formula. So the proof Godel’s Incompleteness Theorem doesn’t doesn’t work if addition is the only allowed mathematical operation. We will use the following strategy.

Recall that we know how to build a finite state machine \( L \) that accepts tuples \( (x, y, z, w) \) properly encoded that satisfy \( (x + y = z) \) and a finite state machine \( M \) that accepts tuples \( (x, y, z, w) \) properly encode that satisfy \( (y + z = w + x) \).

(a) Explain how to construct a finite state machine \( N \) that accepts tuples \( (x, y, z, w) \) properly encoded that satisfy \( \neg(y + z = w + x) \) from finite state machine \( M \).

Hint: This is completely trivial. If you need more than a sentence to explain how to do this, you are not on the right track.

(b) Explain how to construct a finite state machine \( P \) that accepts tuples \( (x, y, z, w) \) properly encoded that satisfy \( (x + y = z) \land \neg(y + z = w + x) \) from finite state machine \( L \) and \( N \).

Hint: The states in \( P \) will be of the form \((l, n)\) where \( l \) is a state in \( L \) and \( n \) is a state in \( N \).
(c) Now consider the quantifiers from the inside out. Explain how to construct a finite state machine $Q$ that accepts tuples $(x, y, z)$ properly encoded with the property that $\exists w (x + y = z) \land \neg (y + z = w + x)$ from finite state machine $P$.

Hint: The states in $Q$ will be subsets of states in $P$.

(d) Explain how to construct a finite state machine $R$ that accepts tuples $(x, y)$ properly encoded with the property that $\forall z \exists w (x + y = z) \land \neg (y + z = w + x)$ from finite state machine $Q$. Hint: The states in $R$ will be subsets of states in $Q$.

(e) Note that the same idea can be used to construct a finite state machine $S$ that accepts strings $x$ properly encoded with the property that $\forall y \forall z \exists w (x + y = z) \land \neg (y + z = w + x)$ from finite state machine $R$.

(f) Give an algorithm that, given as input a finite automata $S$, can determine whether there is any string that $S$ accepts.

(g) Give an algorithm that decides whether a given first order sentence in number theory, that only involves arithmetic operations + and = i and has the quantifiers appearing first, is true. More or less all the main ideas are contained in the previous subproblems. You more or less just need a couple of sentences to put everything together.

Due Friday January 26

9. What happens to the entropy/information of objects that fall into a black hole is an interesting subject. Watch and write a one paragraph summary of http://people.cs.pitt.edu/~kirk/cs1511/BlackHoleEntropy.mp4.

Due Monday January 29

10. Recall the Definition of entropy of a probability distribution $X$:

$$ H(X) = \sum_x P(x) \log(1/P(x)) $$

Here we use the convention that capital letters are probability distributions and lower case letters are a possible outcome. So here $x$ is some possible value of the random variable $X$. So if the setting that $X$ is a random coin flip, then $x$ might either be Heads or Tails.

We now define the conditional entropy of probability distributions $X$ and $Y$ as:

$$ H(X \mid Y) = \sum_y P(y) \sum_x P(x \mid y) \log 1/P(x \mid y) = \sum_{x,y} P(x, y) \log 1/P(x \mid y) $$

Here $P(x \mid y)$ is the probability of event $x$ given event $y$ and $P(x, y)$ the probability of event $x$ and event $y$ both happening. Further define the mutual information between $X$ and $Y$ as:

$$ I(X; Y) = H(X) - H(X \mid Y) $$

Intuitively, the mutual information $I(x; y)$ measures the average reduction in uncertainty (measured in bits) about $x$ that results from learning the value of $y$. 

5
(a) Now assume \(x\) is a bit that a sender wants to send to a receiver over a noisy channel, and let \(y\) be the bit received by the receiver. Because the channel is noisy, \(y\) may not equal \(x\). Let \(X\) be the probability distribution where \(x\) is 0 with probability \(1/3\) and \(x\) is 1 with probability \(2/3\). Assume the noisy channel has the following properties \(P(y = 0 \mid x = 0) = .9, P(y = 1 \mid x = 0) = .1, P(y = 0 \mid x = 1) = .2,\) and \(P(y = 1 \mid x = 1) = .8\). Let \(Y\) be the probability distribution for the received bit.

i. What is \(H(X)\) for this example?

ii. What is the probability distribution \(Y\)? That is, what is the probability that \(y=0\) and what is the probability that \(y=1\)?

iii. What is \(H(Y)\) for this example?

iv. What is \(H(X \mid Y)\) for this example?

v. What is \(H(Y \mid X)\) for this example?

vi. What is \(I(X;Y)\) for this example?

vii. What is \(I(Y;X)\) for this example?

Hint: You should find that \(I(X;Y) = I(Y;X)\).

viii. Restate in plain English what it means that \(I(X;Y) = I(Y;X)\) in this setting.

(b) Now consider arbitrary probability distributions \(X\) for \(x\) and \(Y\) for \(y\).

i. Prove that \(I(X;Y) \geq 0\).

ii. Prove \(I(X;Y) = I(Y;X)\).

FYI, Shannon’s noisy channel coding theorem says: You can get about \(\max_X I(X;Y)\) bits of information through to the receiver for each bit sent.

Due Monday January 29

11. • Say that a string \(x\) of \(n\) bits is semi-incompressible if \(K(x) \geq \sqrt{n}\). Here \(K(x)\) is the Kolmogorov complexity of \(x\). Show that the set of semi-incompressible strings is not computable.

• Show that there are only finitely many incompressible strings that have the property that the number of bits that 0 in the string is equal to the number of bits that are 1 in the string. Recall that a string is incompressible if its Kolmogorov complexity is at least its length.

Hint: You can use without proof the fact that if you flip \(n\) fair independent coins, the probability that the number of heads is equal to the number of tails is approximately \(1/\sqrt{n}\).

• Show that the set of incompressible strings contains no infinite subset that is recursively enumerable.

• Show that the set of compressible strings is recursively enumerable. A string is compressible if it is not incompressible.

Due Wednesday January 31
12. (Extra credit for undergraduates; Required for graduate students) Show that for any \(c > 0\), there exist strings \(x\) and \(y\) exist such that \(K(xy) > K(x) + K(y) + c\). Here \(K(x)\) is the Kolmogorov complexity of \(x\).

Due Wednesday January 31

13. Prove that the following two definitions of \(TIME(T(n))\) are equivalent in the sense that they contain exactly the same languages.

Definition 1: \(TIME(T(n))\) is the set of all languages \(L\) such that there exists a Turing machine \(M\) such that (1) \(M\) accepts \(x\) iff \(x \in L\) and (2) for all but finitely many \(x\), \(M\) on \(x\) halts in \(T(|x|)\) steps.

Definition 2: \(TIME(T(n))\) is the set of all languages \(L\) such that there exists a Turing machine \(N\) and a number \(b\) such that (1) \(N\) accepts \(x\) iff \(x \in L\) and (2) for all \(x\), \(N\) on \(x\) halts within \(b \cdot T(|x|)\) steps.

Hint: You need to show that if you have a Turing machine \(M\) that satisfies the first condition, then you construct from it a Turing machine \(N\) that satisfies the second condition. This direction is pretty straight-forward. And you need to show that if you have a Turing machine \(N\) that satisfies the second condition, then you construct from it a Turing machine \(M\) that satisfies the first condition. This direction is a bit trickier, and you have to show how to speed up any Turing machine by a constant factor. The main insight one needs to accomplish this is to use a larger tape alphabet size.

Due Friday February 2

14. Consider a programming language mini-Java that only has one type of loop, and the number of iterations of the loop must be determined when the loop is first encountered. So a loop statement might look like "Repeat \(x\) times", and the variable \(x\) is evaluated when the statement is first reached. You can assume that the program has a variable \(n\) that is instantiated to the input size when the program starts running (otherwise, you couldn’t even read the input). Note that all mini-Java programs must halt on all inputs. Show by diagonalization that there is a language accepted by a Java program that is not accepted by any mini-Java program.

You can think of any string as a mini-Java program. If the string is not a syntactically correct mini-Java program, then think of it as a mini-Java program that rejects all inputs.

Due Monday February 5

15. (a) Let \(A\) be the language of properly nested parentheses. So for example, \((())\) and \(((()))())\) are in \(A\) but \)( is not in \(A\). Show that \(A\) can be accepted by a log-space Turing machine.

Hint: This is very easy. It is sufficient to store one number between 0 and \(n\) (the number of parentheses in the input) on the work tape while making one pass over the read-only input tape.
(b) Let $B$ be the language of properly nested parentheses and brackets. So for example, $(((\textbf{)})\textbf{(})\textbf{)}\textbf{)}$ is in $B$, but $([])$ is not in $B$. Show that $B$ can be accepted by a log-space Turing machine.

Hint: I don’t see how to do this by making only one pass over the input tape. Start by making sure that $[$ and $]$ don’t appear in the input. Then rule out substrings like $[()\textbf{[}()\textbf{]}\textbf{)}$ and $\textbf{([}())\textbf{]}$. Then continue in this manner.

Due Monday February 5

16. Assume a log-space reduction from a language $A$ to a language $B$.

So more precisely, there is a Turing machine $T$ with three tapes, a read-only input tape, a read/write work tape, and a write-only output tape. $T$ only uses log of the input size many cells on the read/write work tape. Further $T$ never backs up the tape head on the write-only output tape, so the tape head on the write-only tape either stays in position or moves to the right. The machine $T$ has the property that a string $x$ is in $A$ iff the contents of the write tape, when $T$ ends computation on input $x$, is in $B$.

Now show that if there is a log space Turing machine $S$ that accepts $B$, then there is a log space Turing Machine $U$ that accepts $A$.

Hint: I think its harder to understand the question than it is to actually solve the problem once you really understand the question. The machine $U$ has to simulate the computation of both machines $S$ and $T$ while only using log space. Note that this is not trivial because $U$ does not have enough work tape to write down the output of $T$. This is a good problem to show how to make use of the fact that space is reusable.

Due Wednesday February 7

17. (a) Define EXPSPACE to be the set of languages $L$ where there exists a Turing machine $M$, and integer $k$ such that $M$ accepts exactly the language $L$ and using space at most $2^{n^k}$ on all inputs of size $n$. Define a language $C$, and show that $C$ is complete for EXPSPACE under polynomial time reductions.

Hint: This is should be more of less line by line the same logic as showing that PSPACE has a complete language, which we did in class.

(b) Define EXPSPACE to be the set of languages $L$ where there exists a Turing machine $M$, and integer $k$, and an integer $c$ such that $M$ accepts exactly the language $L$ and using space at most $c^{n^k}$ on all inputs of size $n$. Define a language $C$, and show that $C$ is complete for EXPSPACE under polynomial time reductions.

Hint: First ask yourself what issue arises here that didn’t arise in the previous subproblem. Then ask yourself how to address this issue. There are very easy ways to address this issue.

Due Wednesday February 7

18. Read the description of the generalized geography and the proof that it complete for PSPACE under polynomial time reductions at:

(a) Show the Generalized Geography game instance that would result from applying the reduction to the quantified Boolean formula (you can draw this by hand, you don’t need to use LaTeX):

\[ \exists w \forall x \exists y \forall z (x \lor \neg y \lor z) \land (w \lor \neg y \lor \neg z) \land (w \lor y \lor z) \land (y \lor z \lor \neg x) \]

(b) Who has a winning strategy for this instance of the Generalized Geography game, the first player or the second player? Give a brief justification for your claim.

Due Friday February 9

19. (a) Problem 5.9 part (a) from the text. EXACT INDSET is defined in the chapter 5 in the text.
(b) Problem 5.11 from the text. SUCCINCT SET-COVER is defined in the chapter 5 in the text.
(c) Problem 5.13 part (a)
(d) Problem 5.9 part (b)

Hint: Most of these should be pretty straight-forward provided one understands the definition of the polynomial time hierarchy.

Due Monday February 12

20. Problem 5.3 from the text.

Hint: There is nothing deep going on here. The whole issue will be understanding the definitions. Once one understands the definitions, the problem is very straight-forward. You will want to use the fact that 3SAT is complete for NP under Karp reductions (see definition 2.7). Understand why 3SAT and the complement of 3SAT are not obviously reducible to each other using Karp reductions. So this problem is talking about the consequence of 3SAT actually being Karp reducible to its complement. Start by showing that one consequence would be NP=co-NP (note that you show that two sets are equal by showing that each is a subset of the other).

Then I suggest reading section 5.5 first, which is < 2 pages. Then at a high level, you will want to show that if the polynomial time hierarchy collapses at some level, then it collapses at higher levels. Specifically, you want to show that if NP=co-NP then \( \Sigma^p_2 = NP = co-NP \) and \( \Pi^p_2 = NP = co-NP \). Once you have that, the collapse of the higher levels follows from the repeated application of this same argument.

Due Monday February 12

21. (a) Let \( G \) be a Boolean formula with \( n \) variables and \( m \) logical operations. Show how to construct a combinatorial circuit \( C \) with \( n \) input lines and one output line, such that \( C \) outputs a 1 if and only \( G \) is true for that setting of the variables. Further the number of gates in \( C \) should be \( O(n^2m^2) \).
(b) Let $C$ be a combinatorial circuit with input lines $I_1, \ldots, I_n$ and one output line. Further $C$ has $S$ gates. Show how to compute a Boolean formula $G$ over variables $I_1, \ldots, I_n$, and perhaps some other variables, such that $G$ has $O(S^2)$ logical operations (AND, OR, and NOT), and $G$ is satisfiable given a particular setting of the variables $I_1, \ldots, I_n$ if and only if $C$ on those inputs, outputs a 1. So you want $G$ to essentially simulate $C$.

(c) Problem 6.2 from the text.

Due Wednesday February 14

22. Show that there are Boolean functions $f : \{0, 1\}^n \to \{0, 1\}$ that can be computed with $n^4$ gates but not with $n^2$ gates.

Hint: Think about Shannon’s proof of Theorem 6.21 in the text. The trickiest part is probably to show that there are at least say $2^{n^3}$ different Boolean functions that can be computed with $n^4$ gates.

Due Wednesday February 14

23. Problem 6.3 from the text.

Hint: Time Hierarchy Theorem. This is very easy.

Due Friday February 16

24. (a) Problem 6.5 from the text.

(b) Problem 6.6 from the text.

(c) Problem 6.7 from the text.

Hint: See the hints at the back of the book.

Due Friday February 16

25. Problem 6.9 from the text

Due Monday February 19

26. Problem 7.6 from the text. That is, show $\text{ZPP} = \text{RP} \cap \text{coRP}$

Due Monday February 19

27. Problem 7.5 from the text. Explicitly state the probability $p$ that you would use.

Hint: Use the hint at the back of the text, and the law of large numbers:


Don’t worry about making your proof fully formal, concentrate on getting the main idea. I’m mostly interested that you can correctly identify the probability $p$, and can give some intuitive reasoning how to use this probability.

Due Wednesday February 21

28. Define the complexity class $\text{PP}$ to be the collection of languages $L$ for which there is a polynomial $Q$ and there is a probabilistic Turing machine $M$ such that
• $M$ always runs in time at most $Q$ of the input size,
• if $x \in L$ then $M$ accepts with probability strictly greater than $1/2$, and
• if $x \not\in L$ then $M$ rejects with probability strictly greater than $1/2$.

Prove $(\text{NP} \cup \text{co-NP}) \subseteq \text{PP}$.

Hint: There is a very short proof.
Due Wednesday February 21

29. Problem 7.7 from the text.
   Hint: $\text{BP} \cdot \text{NP}$ is defined in Definition 7.17. Recall proof that BPP is in P/poly.
   Due Friday February 23

30. Problem 7.9 from the text.
   Hint: Recall that BPP is in $\Sigma_2^p$ and proof that polynomial time hierarchy is in PSPACE.
   Due Friday February 23

31. Problem 7.8 from the text.
   Hint: Recall the proof that BPP is in $\Sigma_2^p$.
   Due Monday February 26

32. Problem 8.1 from the text. You can use the fact that IP=PSPACE even if we have not proven it in class yet.
   Due Monday March 12

33. Problem 8.2 from the text.
   Due Monday March 12

34. Problem 8.3 from the text.
   Due Monday March 12

35. Problem 8.5 from the text.
   Hint: See the hint at the back of the book. Recall the proof that $\text{BPP} \subseteq \Sigma_2^p$.
   Due Wednesday March 14

36. Show that MAM is a subset of AM. Note that this is a special case of 8.7 in the text, and that there is a hint in the book.

   Further hint: Consider the MAM protocol. Assume the protocol guarantees the verifier makes the wrong decision with probability at most $1/3$. Assume the first message from the prover has $m$ bits. Explain how to modify the protocol to get another MAM protocol where the verifier is wrong with probability at most $1/4^m$. You can use the following Chernoff bound: if you have $k$ independent (unfair) coin flips that each come up heads with probability $p$, then the expect number of heads is $kp$, and the probability
that the number of heads is either more than 10% more or less than 10% less is at most $2 \cdot e^{-p^2k/3}$. Then use the hint in the back of the book and a union bound.

Due Friday March 16

37. Consider the interactive protocol for the language TQBF, of true quantified Boolean formulas. Consider the following formula $F$:

$$\exists x \forall y \exists z \left( x \lor y \lor \neg z \right) \land \left( \neg x \lor \neg y \lor z \right)$$

(a) First consider the protocol without linearization.
   i. What is the integer $S$ and polynomial $s(x)$ that Merlin sends in the first round? Show your work.
   ii. Assuming that Arthur responds by sending the message $1/3$ to Merlin, what is the polynomial that Merlin sends in his second message to Arthur? Show your work.
   iii. Arthur checks this second polynomial to see if it has some property, what property is this?

(b) Now consider the protocol with linearization.
   i. What is the integer $S$ and polynomial $s(x)$ that Merlin sends in the first round? Show your work.
   ii. Assuming that Arthur responds by sending the message $1/3$ to Merlin, what is the polynomial that Merlin sends in his second message to Arthur? Show your work.
   iii. Arthur checks this second polynomial to see if it has some property, what property is this?

Due Friday March 16

38. (Required for graduate students. Extra credit for undergraduate students) This homework is to be done individually, not in your group. Read the history of the IP=PSPACE result given in http://www.cs.pitt.edu/~kirk/cs2110/BabaiIPStory.pdf. Write a paragraph explaining what you think was the right way to handle the publication issue. That is, “What papers should have come out of this?” and “Who should the authors have been on each paper?”

Due Monday March 19

39. Problem 8.8 from the text.

Due Monday March 19

40. Problem 9.3 from the text

   Hint: This is super easy. There is more or less a one line proof.

Due Monday March 19
41. Problem 9.5 from the text
   Hint: Show that the problem of inverting a one-way function can be solved by a non-deterministic polynomial-time machine. You may assume (without any real loss of generality) that for all inputs, the output of the one way function contains at least as many bits as the input.
   Hint: This is super easy. There is more or less a one line proof.
   Due Monday March 19

42. Problem 9.2 from the text
   Hint: First do the proof for \( m = 2 \) and \( n = 1 \). It will be seem way easier, but the general proof is more or less the same as the proof when \( m = 2 \) and \( n = 1 \).
   Due Wednesday March 21

43. Problem 9.11 from the text
   Hint: This is relatively straightforward.
   Due Wednesday March 21.

44. Problem 9.6 from the text.
   Hint: Part (a) is easy using a trick we have seen before. For part (b) show how to use a polynomial time algorithm for inverting \( f_U \) to build a polynomial time algorithm for inverting \( f \).
   Due Friday March 23

45. Problem 9.9 from the text.
   Hint: There are hints in the back. Half of the work here will be in understanding the definitions.
   Due Friday March 23

46. Read and understand the protocol in problem exercise 9.17 part b from the text.
   (a) Write a one paragraph summary of the protocol in your own words.
   (b) Then write a paragraph explaining intuitively what it means for an interactive proof to be computationally zero knowledge.
   (c) Then write a paragraph to explain why it is intuitively believable why this protocol is computationally zero knowledge.
   You do not have to formally prove anything.
   Due Monday March 26

47. It is shown in https://en.wikipedia.org/wiki/Fredkin_gate that the Fredkin gate is universal and reversible. Similarly show that the Toffoli gate, see https://en.wikipedia.org/wiki/Toffoli_gate for a description, is universal and reversible.
   Due Monday March 26
48. Recall the half-silvered mirror experiment, which was essentially a Hadamard gate, followed by a "not", followed by a Hadamard gate. If photon came in horizontally, then came out horizontally, and if photon came in vertically, then it left vertically.

(a) Calculate the outgoing state of the photon if the incoming state is in the superposition state: \(a|H\rangle + b|V\rangle\). Here \(|H\rangle\) represents horizontal and \(|V\rangle\) represents vertical.

You can assume that
\[
|H\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}
\]
and
\[
|V\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\]

You can assume that the half-silvered mirror implements the 1-bit Hadamard operation
\[
\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}
\]

You can assume that the full silvered mirror implements the not operation
\[
|V\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}
\]

(b) What is the probability that an observer sees the photon come out of the last half-silvered mirror horizontally?

(c) What is the probability that an observer sees the photon come out of the last half-silvered mirror horizontally?

Due Monday March 26

49. Problem 10.6 from the text.

Due Wednesday March 28

50. (a) Work out in detail the probability that Alice and Bob win in the case that \(x = y = 1\) in the EPR experiment in the book. Show every step of your calculations, and give lots of explanation. You can assume without loss of generality that the order of events is Alice rotates, Bob rotates, Alice measures, and then finally Bob measures.

(b) Consider the following protocol for Alice and Bob (which I had on the board at some point).

Alice: If \(x = 0\) then \(a = 0\) else \(a\) is the result of measuring Alice’s entangled bit after rotating it by \(\pi/8\)

Bob: If \(y = 0\) then \(b = 0\) else \(b\) is the result of measuring Bob’s entangled bit after rotating it by \(-\pi/8\).
Calculate the probability of Alice and Bob winning with this protocol (you can reuse calculations for the protocol in the textbook as appropriate). Is this probability more, less or the same as for the probability of Alice and Bob winning for the protocol in the textbook?

Due Wednesday March 28

51. Assume you have two qubits in the state $a \ket{00} + b \ket{01} + c \ket{10} + d \ket{11}$.

(a) Show the result of applying the 2 bit Hadamard operation to these bits.
(b) Show the result of applying the 1 bit Hadamard operation to the first qubit.
(c) Show the result of applying the 1 bit Hadamard operation to the first qubit followed by applying the 1 bit Hadamard operation to the second bit.

Hint: The first and the third answers should be the same.

Due Friday March 30

52. Consider the same set up as the parity game. Alice and Bob split two entangled bits $a$ and $b$ in state $\ket{00}/\sqrt{2} + \ket{11}/\sqrt{2}$. Alice and Bob are then split up. Alice is then given 2 classical bits $x$ and $y$. Depending on the value of $x$, and $y$, Alice performs some operations on qubit $a$ and then sends qubit $a$ to Bob. The 4 possible states of the sent qubit will be the 4 Bell states: $1/\sqrt{2} \ket{0} + 1/\sqrt{2} \ket{1}$, $1/\sqrt{2} \ket{0} - 1/\sqrt{2} \ket{1}$, $-1/\sqrt{2} \ket{0} + 1/\sqrt{2} \ket{1}$, and $-1/\sqrt{2} \ket{0} - 1/\sqrt{2} \ket{1}$. From this single qubit, Bob determines with certainty the two classical bits $x$ and $y$ from his entangled bit and the sent qubit.

(a) Explain what operations Alice can perform to change the state of qubit $a$ to each of the 4 Bell states depending on the 4 possible values of the classical bits $x$ and $y$.
(b) State what the state of qubits $a$ and $b$ are when Bob receives them. This state will depend on the value of the classical bits $x$ and $y$.
(c) Explain how Bob can retrieve the bits $x$ and $y$ from operations and measurements on qubits $a$ and $b$. Hint: Use Hadamard operations. Call this a Bell measurement.

Due Friday March 30

53. The goal of this problem is to find a way to transmit information about a qubit by sending two classical bits. Alice and Bob split up entangled bits $a$ and $b$ in state $\ket{00}/\sqrt{2} + \ket{11}/\sqrt{2}$. Assume that now Alice is given qubit $x$. So $x$ is in some unknown superposition $\alpha \ket{0} + \beta \ket{1}$ between states $\ket{0}$ and $\ket{1}$. Alice and Bob then follow the following informally stated protocol:

- Alice performs a Bell measurement on qubit $a$ and qubit $x$, which you developed in the last homework problem. This results in 4 possible outcomes.
- Alice sends Bob two classical bits describing the outcome of this measurement.
• As a result of Alice’s measurement, qubit $b$ is in one of four possible states. Of these four possible states, one is identical to the original quantum state, and the other three are closely related. Which of these four possibilities actually obtains is encoded in the two classical bits. Knowing this, the qubit $b$ can be modified in one of three ways, or not at all, to result in a qubit identical to $x$.

Explain how to implement the last step, and prove that it works. To prove it works you just need to show by calculations that the final state of qubit $b$ is the initial state of qubit $x$.

Alice now performs the following reversible operation on $x$, if $a = 1$ then negate $x$. Alice then runs qubit $a$ through a Hadamard gate. Alice now measures the current values of $a$ and $x$, and sends these two classical bits to Bob. Explain what the state of all the particles $a$, $b$, and $x$ is after each of Alice’s operations. Then explain how Bob can use the two classical particles to change the state of $b$ to the original state of $x$. Note that when Alice measures $a$ and $x$, then she can no longer recover the original state of qubit $x$.

Due Monday April 2

54. Problem 10.5 from the text
Due Monday April 2

55. Read Scott Aaronson’s intuition for Shor’s quantum polynomial time factoring algorithm, found here: [http://www.scottaaronson.com/blog/?p=208](http://www.scottaaronson.com/blog/?p=208). You don’t have to do anything. But this is now fair game for a B question on midterm 2.

56. Problem 11.2 from the text. This is very easy.
Due Wednesday April 4

57. Problem 11.7 from the text. This should be pretty straightforward. Just convert the interactive proof for the permanent from chapter 8 as the interactive proof for graph non-isomorphism was converted in example 11.7.
Due Wednesday April 4

58. Problem 11.16 from the text. Note that “equation” in this context means that the relation is equality. So one possible equation might be $1.5x + 2y + 2.5z = 10.5$. Note that rational solutions are allowed. So $x = 2$, $y = .5$ and $z = 3$ is an allowable solution.
Hint: There is a not too complicated reduction from MAXSAT.
Due Wednesday April 4

59. (a) Give a solution $u$ to the following instance of QUADEQ:

\[
\begin{align*}
  u_1u_1 + u_1u_2 + u_3u_3 &= 0 \\
  u_1u_3 + u_2u_2 &= 1 \\
  u_1u_1 + u_1u_2 &= 1 \\
  u_2u_2 + u_2u_3 &= 0
\end{align*}
\]
\[ u_2u_3 + u_1u_3 + u_1u_2 = 1 \]
That is give a \( u_1 \), \( u_2 \), and \( u_3 \) that satisfy these equations.

(b) Write a program in Java, or your favorite programming language if you don’t like Java, that takes as input a solution \( u \) to some instance of QUADEQ and produces the probabilistically checkable proof constructed in in the proof of Theorem 11.19 in the text (and the proof we did in class). Turn in your code. You code can assume that \( u \) is not more than 10 bits. Input and output conventions don’t really matter. Turn in your code.

(c) Give the output of your program for the \( u \) you found in the first subproblem. Hint: You output should have 520 bits.

(d) Specify which of these 520 bits would give you the value of: \( u_1u_2 + u_2u_3 + u_3u_3 \).
So I’m looking for a number between 1 and 520 inclusive. Show your work, as the answer may depend upon the convention on ordering the bits.

Due Friday April 6

60. Problem 11.9 from the text
Due Monday April 9

61. Problem 13.2 from the text
Due Wednesday April 11

62. Problem 13.4 from the text
Due Wednesday April 11

63. Problem 13.13 from the text. Prove an \( O(\log^2 n) \) upper bound and an \( \Omega(\log n) \) lower bound on the communication complexity.
Due Friday April 13

64. Problem 13.19 from the text
Due Friday April 13