1 Problem 6.2

(a) Prove that for every integer n, there exists a coloring of the edges of the complete graph $K_n$ by two colors so that the total number of monochromatic copies of $K_4$ is at most $\binom{n}{4} 2^{-5}$.

(b) Give a randomized algorithm for finding a coloring with at most $\binom{n}{4} 2^{-5}$ monochromatic copies of $K_4$ that runs in expected time polynomial in $n$.

(c) Show how to construct such a coloring deterministically in polynomial time using the methods of conditional expectations.

Proof:

(a) Each edge has 1/2 probability to be colored in each of the two colors. There are 6 edges in each $K_4$, so the probability that a $K_4$ is monochromatic is $2 \cdot \frac{1}{2^6} = 2^{-5}$.

The expected number of monochromatic $K_4$ is thus $\binom{n}{4} \cdot 2^{-5}$, so that there exists such coloring such that the number of monochromatic $K_4$ is at most $\binom{n}{4} 2^{-5}$.

(b) By flipping a coin, if it is head, we use color 1, otherwise we use color 2. Assuming we have probability $p$ to get such a coloring, then:

$$\binom{n}{4} 2^{-5} \geq (1 - p) \left( \binom{n}{4} 2^{-5} + 1 \right)$$

So,

$$1 - p \leq \frac{\binom{n}{4}/32}{\binom{n}{4}/32 + 1}$$

$$p \geq 1 - \frac{\binom{n}{4}/32}{\binom{n}{4}/32 + 1}$$

$$= \frac{1}{\binom{n}{4}/32 + 1} \quad (1)$$

Simply repeat the process, with success probability of $p$. The expected number of trials before a success is thus $1/p$, where

$$\frac{1}{p} = \frac{\binom{n}{4}}{32 + 1} = 1 + O(n^4) = O(n^4) \quad (2)$$

(c)
2 Problem 6.14

Consider a graph in $G_{n,p}$, with $p = 1/n$. Let $X$ be the number of triangles in the graph, where a triangle is a clique with three edges. Show that

$$ Pr(X \geq 1) \leq 1/6 $$

and that

$$ \lim_{n \to \infty} Pr(X \geq 1) \geq 1/7 \quad (3) $$

( Hint: Use the conditional expectation inequality.

Proof:

By using the conditional expectation inequality, we have

$$ Pr(X \geq 1) = Pr(X > 0) = \sum_{i=1}^{\binom{n}{3}} \frac{Pr(X_i = 1)}{E[X|X_i = 1]} \quad (4) $$

For any triangle, $Pr(X_i = 1) = p^3$, and

$$ E[X|X_i = 1] = 1 + \binom{n-3}{3} p^3 + \binom{n-3}{2} p^3 + \binom{n-3}{1} p^2 \quad (5) $$

So,

$$ Pr(X \geq 1) = \sum_{i=1}^{\binom{n}{3}} \frac{p^3}{1 + \binom{n-3}{3} p^3 + \binom{n-3}{2} p^3 + \binom{n-3}{1} p^2} \leq \frac{\binom{n}{3} p^3}{1 + \binom{n-3}{3} p^3 + \binom{n-3}{2} p^3 + \binom{n-3}{1} p^2} \leq \frac{n^3}{6} p^3 \quad (6) $$

Since $p = 1/n$, we have $Pr(X \geq 1) \leq (n^3 p^3)/6$. 
We also have:

\[
\lim_{n \to \infty} Pr(X \geq 1) = \lim_{n \to \infty} \frac{\binom{n}{3} p^3}{1 + \binom{n-3}{3} p^3 + \binom{n-3}{2} p^3 + \binom{n-3}{1} p^2}
\]

\[
= \lim_{n \to \infty} \frac{\binom{n}{3}/n^3}{1 + \binom{n-3}{3}/n^3 + \binom{n-3}{2}/n^3 + \binom{n-3}{1}/n^3}
\]

\[
= \lim_{n \to \infty} \frac{n(n-1)(n-2)/(6n^3)}{1 + (n-3)(n-4)(n-5)/(6n^3) + (n-3)(n-4)/(2n^3) + (n-3)/n^3}
\]

\[
= \lim_{n \to \infty} \frac{1/6}{1 + 1/6 + 0 + 0}
\]

\[
= \frac{1/6}{7/6}
\]

\[
= \frac{1}{7}
\]  

(7)

3 Problem 6.16

Use the lovatz local lemma to show that if

\[
4 \cdot \binom{k}{2} \cdot \binom{n}{k-2} 2^{1-\left(\frac{k}{2}\right)} \leq 1
\]

then it is possible to color the edges of \(K_n\) with two colors so that is has no monochromatic \(K_k\) subgraph.

Proof:

Using lovatz local lemma, we need to \(4dp \leq 1\).

Define the events set \(E\) as

\[
E = \{E_i \mid E_i \in K_k, E_i \text{ is monochromatic}\}
\]

By flipping coins, we can color the graph with two colors, each color has the same probability of \(1/2\). The probability for each \(E_i\) is thus \(2 \cdot (\frac{1}{2})^\left(\frac{k}{2}\right) = 2^{1-\left(\frac{k}{2}\right)}\), since there are \(\binom{k}{2}\) edges in each \(K_k\).

Next we need to prove that \(d = \binom{k}{2} \binom{n}{k-2}\).

Two \(K_k\) are dependent means they should have at least two common edges, or say three common vertices. To bound the degree of dependence, we do the following. Choose one edge from the current \(K_k\), choose one or more vertices from this \(K_k\), and the rest from all vertices not in this \(K_k\). Any such a selection is an instance of choose \(k-2\) vertices from all the \(n\) vertices. So the degree of dependence is bounded by

\[
d = \binom{k}{2} \binom{n}{k-2}
\]

We have \(4dp \leq 1\), so we have

\[
Pr\left(\bigcup_{i=1}^{n} \bar{E}_i\right) > 0
\]  

(11)
which means that it is possible to color the graph such that neither $K_k$ is monochromatic.

For any one $K_k$, to construct a graph which is dependent on it, we can choose 2 vertices from this graph, and the rest from all vertices which are not in this $K_k$. Any such an instance must be an instance that