Discrete Structures for Computer Science

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Lecture #10: Sequences and Summations

Based on materials developed by Dr. Adam Lee
Today’s Topics

Sequences and Summations

- Specifying and recognizing sequences
- Summation notation
- Closed forms of summations
- Cardinality of infinite sets
Sequences are ordered lists of elements

**Definition:** A sequence is a function from the set of integers to a set \( S \). We use the notation \( a_n \) to denote the image of the integer \( n \). \( a_n \) is called a term of the sequence.

**Examples:**

- \( 1, 3, 5, 7, 9, 11 \)  
  A sequence with 6 terms
- \( 1, 1/2, 1/3, 1/4, 1/5, \ldots \)  
  An infinite sequence

**Note:** The second example can be described as the sequence \( \{a_n\} \) where \( a_n = 1/n \)
What makes sequences so special?

Question: Aren’t sequences just sets?

Answer: The elements of a sequence are members of a set, but a sequence is ordered, a set is not.

Question: How are sequences different from ordered n-tuples?

Answer: An ordered n-tuple is ordered, but always contains n elements. Sequences can be infinite!
Some special sequences

**Geometric progressions** are sequences of the form \( \{ar^n\} \) where \( a \) and \( r \) are real numbers.

**Examples:**
- \( 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots \)
- \( 1, -1, 1, -1, 1, -1, \ldots \)

**Arithmetic progressions** are sequences of the form \( \{a + nd\} \) where \( a \) and \( d \) are real numbers.

**Examples:**
- \( 2, 4, 6, 8, 10, \ldots \)
- \( -10, -15, -20, -25, \ldots \)
Sometimes we need to figure out the formula for a sequence given only a few terms

Questions to ask yourself:

1. Are there runs of the same value?
2. Are terms obtained by multiplying the previous value by a particular amount? (Possible geometric sequence)
3. Are terms obtained by adding a particular amount to the previous value? (Possible arithmetic sequence)
4. Are terms obtained by combining previous terms in a certain way?
5. Are there cycles amongst terms?
What are the formulas for these sequences?

**Problem 1:** 1, 5, 9, 13, 17, ...

**Problem 2:** 1, 3, 9, 27, 81, ...

**Problem 3:** 2, 3, 3, 5, 5, 5, 7, 7, 7, 7, 11, 11, 11, 11, 11, ...

**Problem 4:** 1, 1, 2, 3, 5, 8, 13, 21, 34, ...

This is called the Fibonacci sequence.
Sometimes we want to find the sum of the terms in a sequence.

**Summation notation** lets us compactly represent the sum of terms $a_m + a_{m+1} + ... + a_n$.

**Example:** $\sum_{1 \leq i \leq 5} i^2 = 1 + 4 + 9 + 16 + 25 = 55$
The usual laws of arithmetic still apply

\[
\sum_{j=1}^{n} (ax_j + by_j + cz_j) = a \sum_{j=1}^{n} x_j + b \sum_{j=1}^{n} y_j - c \sum_{j=1}^{n} z_j
\]

Constant factors can be pulled out of the summation

A summation over a sum (or difference) can be split into a sum (or difference) of smaller summations

Example:
- \[ \sum_{1 \leq j \leq 3} (4j + j^2) = \]
- \[ 4\sum_{1 \leq j \leq 3} j + \sum_{1 \leq j \leq 3} j^2 = \]
Example sums

Example: Express the sum of the first 50 terms of the sequence $1/n^2$ for $n = 1, 2, 3, ...$

Answer: $\sum_{j=1}^{50} \frac{1}{j^2}$

Example: What is the value of $\sum_{k=4}^{8} (-1)^k$

Answer: $\sum_{k=4}^{8} (-1)^k = \sum_{k=4}^{8} (-1)^k = \sum_{k=4}^{8} (-1)^k = \sum_{k=4}^{8} (-1)^k = \sum_{k=4}^{8} (-1)^k = \sum_{k=4}^{8} (-1)^k = \sum_{k=4}^{8} (-1)^k = \sum_{k=4}^{8} (-1)^k$
We can also compute the summation of the elements of some set

**Example:** Compute \[ \sum_{s \in \{0,2,4,6\}} (s + 2) \]

**Answer:** \( (0 + 2) + (2 + 2) + (4 + 2) + (6 + 2) = 20 \)

**Example:** Let \( f(x) = x^3 + 1 \). Compute \[ \sum_{s \in \{1,3,5,7\}} f(s) \]

**Answer:** \( f(1) + f(3) + f(5) + f(7) = 2 + 28 + 126 + 344 = 500 \)
Sometimes it is helpful to shift the index of a summation. This is particularly useful when combining two or more summations. For example:

\[ S = \sum_{j=1}^{10} j^2 + \sum_{k=2}^{11} (2k - 1) \]

Let \( j = k - 1 \)

Need to add 1 to each \( j \)
Summations can be nested within one another

Often, you’ll see this when analyzing nested loops within a program (i.e., CS 1501/1502)

**Example:** Compute \[ \sum_{j=1}^{4} \sum_{k=1}^{3} (jk) \]

**Solution:**

\[
\sum_{j=1}^{4} \sum_{k=1}^{3} (jk) = \sum_{j=1}^{4} (j + 2j + 3j) = \sum_{j=1}^{4} 6j = 6 + 12 + 18 + 24 = 60
\]
Group work!

Problem 1: What are the formulas for the following sequences?
   a. 3, 6, 9, 12, 15, ...
   b. 1/3, 2/3, 4/3, 8/3, ...
   c. 1, 2, 2, 3, 4, 4, 5, 6, 6, 7, 8, 8, ...

Problem 2: Compute the following summations:

   a. \[ \sum_{k=1}^{5} (k + 1) \]
   b. \[ \sum_{k=0}^{8} (2^{k+1} - 2^k) \]
Computing the sum of a geometric series by hand is time consuming...

Would you really want to calculate \( \sum_{j=0}^{20} (6 \times 2^j) \) by hand?

Fortunately, we have a closed-form solution for computing the sum of a geometric series:

\[
\sum_{j=0}^{n} ar^j = \begin{cases} 
\frac{ar^{n+1}-a}{r-1} & \text{if } r \neq 1 \\
(n+1)a & \text{if } r = 1 
\end{cases}
\]

So, \( \sum_{j=0}^{20} (6 \times 2^j) = \frac{6 \times 2^{21} - 6}{2 - 1} = 12,582,906 \)

Why?
Proof of geometric series closed form
There are other closed form summations that you should know

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We can use the notion of sequences to analyze the cardinality of infinite sets

**Definition:** Two sets A and B have the same cardinality if and only if there is a one-to-one correspondence from A to B.

**Definition:** A finite set or a set that has the same cardinality as the natural numbers is called countable. A set that is not countable is called uncountable.

**Implication:** Any sequence \( \{a_n\} \) ranging over the natural numbers is countable.
Show that the set of even positive integers is countable

**Proof #1 (Graphical):** We have the following 1-to-1 correspondence between the natural numbers and the even positive integers:

So, the even positive integers are countable. ✷

**Proof #2:** We can define the even positive integers as the sequence \{2k\} for all \(k \in \mathbb{N}\), so it has the same cardinality as \(\mathbb{N}\), and is thus countable. ✷
Is the set of all rational numbers countable?

Perhaps surprisingly, yes!

This yields the sequence $1/1, 1/2, 2/1, 3/1, 1/3, \ldots$, so the set of rational numbers is countable. ☑
Is the set of real numbers countable?

No, it is not. We can prove this using a proof method called diagonalization, invented by Georg Cantor.

**Proof:** Assume that the set of real numbers is countable. Then the subset of real numbers between 0 and 1 is also countable, by definition. This implies that the real numbers can be listed in some order, say, \( r_1, r_2, r_3 \) ....

Let the decimal representation these numbers be:

\[
\begin{align*}
    r_1 &= 0.d_{11}d_{12}d_{13}d_{14}... \\
    r_2 &= 0.d_{21}d_{22}d_{23}d_{24}... \\
    r_3 &= 0.d_{31}d_{32}d_{33}d_{34}... \\
    &\vdots
\end{align*}
\]

Where \( d_{ij} \in \{0,1,2,3,4,5,6,7,8,9\} \ \forall \ i,j \)
Now, form a new decimal number \( r = 0.d_1d_2d_3\ldots \) where \( d_i = 0 \) if \( d_{ii} = 1 \), and \( d_i = 1 \) otherwise.

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Example:

\[
\begin{align*}
r_1 &= 0.123456\ldots \\
r_2 &= 0.234524\ldots \\
r_3 &= 0.631234\ldots \\
\ldots \\
r &= 0.010\ldots
\end{align*}
\]

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Note that the \( i^{\text{th}} \) decimal place of \( r \) differs from the \( i^{\text{th}} \) decimal place of each \( r_i \), by construction. Thus \( r \) is not included in the list of all real numbers between 0 and 1. This is a contradiction of the assumption that all real numbers between 0 and 1 could be listed. Thus, not all real numbers can be listed, and \( \mathbb{R} \) is uncountable. \( \Box \)
Final thoughts

- Sequences allow us to represent (potentially infinite) ordered lists of elements

- Summation notation is a compact representation for adding together the elements of a sequence

- We can use sequences to help us compare the cardinality of infinite sets

Next time:
  - Integers and division (Section 4.1)