Discrete Structures for Computer Science

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Lecture #4: Logic Puzzles and Propositional Equivalence

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Today's topics

- Logic puzzles
- Propositional equivalences

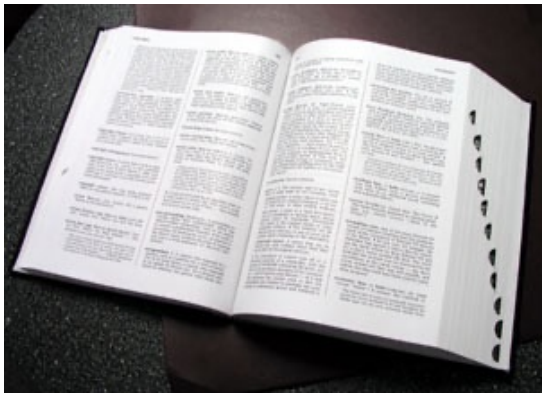
A technical support conundrum

Alice and Bob are technical support agents. If an agent is having a bad day, they will always lie to you. If an agent is having a good day, they will always tell you the truth. Alice tells you that Bob is having a bad day. Bob tells you that he and Alice are both having the same type of day. Can you trust the advice you receive from Alice during your call?

How do we solve this type of problem?

Solving logic puzzles systematically

Step 1: Identify rules and constraints



Step 2: Assign propositions to key concepts

Step 3: Make assumptions and reason logically!

Technical support revisited

Alice and Bob are technical support agents. If an agent is having a bad day, they will always lie to you. If an agent is having a good day, they will always tell you the truth. Alice tells you that Bob is having a bad day. Bob tells you that he and Alice are both having the same type of day. Can you trust the advice you receive from Alice during your call?

Step 1: Identify the rules of the puzzle

Step 2: Assign propositions to the key concepts in the puzzle

Step 3: Make assumptions and reason logically

Another example

Consider a group of friends: Fredrik, Anuradha, and Cai. If Fredrik is not the oldest, then Anuradha is. If Anuradha is not the youngest, then Cai is the oldest. Determine the relative ages of Fredrik, Anuradha, and Cai.

Propositions:

Rules:

Step 3: Make assumptions and reason logically

Sometimes no solution is a solution!

Alice and Bob are technical support agents. Alice says, "I am having a good day." Bob says, "I am having a good day." Can you trust either Alice or Bob?

Step 1: Identify rules

Step 2: Assign propositions

Step 3: Make assumptions and reason logically

In-class exercises

Problem 1: Alice and Bob are technical support agents working to fix your computer. Alice tells you that Bob is having a bad day today and that you should expect a long wait before your computer is fixed. Bob tells you not to worry, Alice is just having a bad day—your computer will be ready in no time.

Question: Can you draw any conclusions about when your computer will be fixed? If so, what can you learn?

Propositional equivalences: preliminaries

Definition: A tautology is a compound proposition that is always true, regardless of the truth values of the propositions occurring within it.

Definition: A contradiction is a compound proposition that is always false, regardless of the truth values of the propositions occurring within it.

Definition: A contingency is a compound proposition whose truth value is dependent on the propositions occurring within it.

Examples

Are the following compound propositions tautologies, contradictions, or contingencies?

- $p \vee \neg p$ tautology
- $\neg p \wedge p$ contradiction
- $p \vee q$ contingency

What are logical equivalences and why are they useful?

Definition: Compound propositions p and q are logically equivalent exactly when $p \leftrightarrow q$ is a tautology. The notation $p \equiv q$ means that p and q are logically equivalent.

Logical equivalences are extremely useful!

- Aid in the construction of proofs
- Allow us to simplify compound propositions

Example: $p \rightarrow q \equiv \neg p \vee q$

How do we prove this type of statement?

Proving logical equivalences the "easy" (but tedious) way

We can prove simple logical equivalences using our good friend the truth table!

Prove: $p \rightarrow q \equiv \neg p \vee q$

p	q	$\neg p$	$\neg p \vee q$	$p \rightarrow q$
T	T			
T	F			
F	T			
F	F			

DeMorgan's laws allow us to distribute negation over compound propositions

Two laws:

- $\neg(p \vee q) \equiv \neg p \wedge \neg q$
- $\neg(p \wedge q) \equiv \neg p \vee \neg q$

If "p or q" isn't true, then neither p nor q is true

If "p and q" isn't true, then at least one of p or q is false

Prove: $\neg(p \vee q) \equiv \neg p \wedge \neg q$

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
T	T					
T	F					
F	T					
F	F					

Using DeMorgan's laws

Use DeMorgan's laws to negate the following expressions:

- "Tito is wearing blue pants and a sweatshirt"
  - $b \wedge s$
  - $\neg(b \wedge s) \equiv \neg b \vee \neg s$
  - Tito is not wearing blue pants or is not wearing a sweatshirt
- "I will drive or I will walk"
  - $d \vee w$
  - $\neg(d \vee w) \equiv \neg d \wedge \neg w$
  - I will not drive and I will not walk

In-class exercises

Problem 2: Prove that $\neg(p \wedge q)$ and $\neg p \vee \neg q$ are logically equivalent, i.e., $\neg(p \wedge q) \equiv \neg p \vee \neg q$. This is the second DeMorgan's law.

Problem 3: Use DeMorgan's laws to negate the following propositions:

- Today I will go running or ride my bike
- Tom likes both pizza and beer

One more on Top Hat

Sometimes using truth tables to prove logical equivalencies can become cumbersome

Recall that for an equivalence with *n* propositions, we need to build a truth table with 2^n rows

- Fine for tables with n = 2, 3, or 4
- Consider n = 30—we would need 1,073,741,824 rows in the truth table!

Another option: Direct manipulation of compound propositions using known logical equivalencies

There are many useful logical equivalences

Equivalence	Name

More useful logical equivalences

Equivalence	Name

These, and more equivalencies, are in the book!

Prove that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology

Prove: $(p \rightarrow q) \vee (p \rightarrow r) \equiv p \rightarrow (q \vee r)$

Final Thoughts

- Logic can help us solve real world problems and play challenging games
- Logical equivalences help us simplify complex propositions and construct proofs
  - More on proofs later in the course
- Next:
  - Predicate logic and quantification
  - Please read section 1.4